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23

Longitudinal and Transverse Vibrations

Features (Main)

23.1. Introduction

When elastic bodies such as a spring, a beam and a shaft are displaced from the equilibrium position by the application of external forces, and then released, they execute a ***vibratory motion.*** This is due to the reason that, when a body is displaced, the internal forces in the form of elastic or strain energy are present in the body. At release, these forces bring the body to its original position. When the body reaches the equilibrium position, the whole of the elastic or strain energy is converted into kinetic energy due to which the body continues to move in the opposite direction. The whole of the kinetic energy is again converted into strain energy due to which the body again returns to the equilibrium position. In this way, the vibratory motion is repeated indefinitely.

23.2. Terms Used in Vibratory Motion

The following terms are commonly used in connection with the vibratory motions :

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1. *Period of vibration or time period.* It is the time interval after which the motion is repeated itself. The period of vibration is usually expressed in seconds.

2. *Cycle.* It is the motion completed during one time period.

3. *Frequency.* It is the number of cycles described in one second. In S.I. units, the frequency is expressed in hertz (briefly written as Hz) which is equal to one cycle per second.

23.3. Types of Vibratory Motion

The following types of vibratory motion are important from the subject point of view :

1. *Free or natural vibrations.* When no external force acts on the body, after giving it an initial displacement, then the body is said to be under ***free or natural vibrations.*** The frequency of the free vibrations is called ***free or natural frequency.***

2. *Forced vibrations.* When the body vibrates under the influence of external force, then the body is said to be under ***forced vibrations.*** The external force applied to the body is a periodic disturbing force created by unbalance. The vibrations have the same frequency as the applied force.

Note : When the frequency of the external force is same as that of the natural vibrations, resonance takes place.

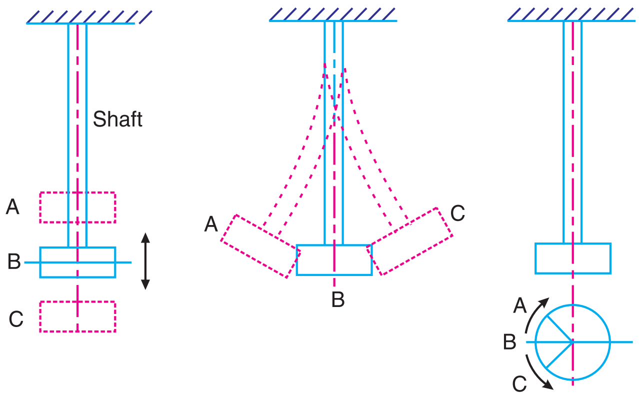
3. *Damped vibrations.* When there is a reduction in amplitude over every cycle of vibration, the motion is said to be ***damped vibration.*** This is due to the fact that a certain amount of energy possessed by the vibrating system is always dissipated in overcoming frictional resistances to the motion.

23.4. Types of Free Vibrations

The following three types of free vibrations are important from the subject point of view :

1. Longitudinal vibrations, **2.** Transverse vibrations, and **3.** Torsional vibrations.

Consider a weightless constraint (spring or shaft) whose one end is fixed and the other end carrying a heavy disc, as shown in Fig. 23.1. This system may execute one of the three above mentioned types of vibrations.



B = Mean position ; *A* and *C* = Extreme positions.

(*a*) Longitudinal vibrations. (*b*) Transverse vibrations. (*c*) Torsional vibrations.

Fig. 23.1. Types of free vibrations.

1. *Longitudinal vibrations.* When the particles of the shaft or disc moves parallel to the axis of the shaft, as shown in Fig. 23.1 (*a*), then the vibrations are known as ***longitudinal vibrations.*** In this case, the shaft is elongated and shortened alternately and thus the tensile and compressive stresses are induced alternately in the shaft.

2. *Transverse vibrations.* When the particles of the shaft or disc move approximately perpendicular to the axis of the shaft, as shown in Fig. 23.1 (*b*), then the vibrations are known as ***transverse vibrations.*** In this case, the shaft is straight and bent alternately and bending stresses are induced in the shaft.

Bridges should be built taking vibrations into account.

3. *Torsional vibrations**. When the particles of the shaft or disc move in a circle about the axis of the shaft, as shown in Fig. 23.1 (*c*), then the vibrations are known as ***torsional vibrations.*** In this case, the shaft is twisted and untwisted alternately and the torsional shear stresses are induced in the shaft.

Note : If the limit of proportionality (*i.e.* stress proportional to strain) is not exceeded in the three types of vibrations, then the restoring force in longitudinal and transverse vibrations or the restoring couple in torsional vibrations which is exerted on the disc by the shaft (due to the stiffness of the shaft) is directly proportional to the displacement of the disc from its equilibrium or mean position. Hence it follows that the acceleration towards the equilibrium position is directly proportional to the displacement from that position and the vibration is, therefore, simple harmonic.

23.5. Natural Frequency of Free Longitudinal Vibrations

The natural frequency of the free longitudinal vibrations may be determined by the following three methods :

1. *Equilibrium Method*

Consider a constraint (*i.e.* spring) of negligible mass in an unstrained position, as shown in Fig. 23.2 (*a*).

Let s = Stiffness of the constraint. It is the force required to produce unit displacement in the direction of vibration. It is usually expressed in N/m.

m = Mass of the body suspended from the constraint in kg,

W = Weight of the body in newtons = $m.g$,

* The torsional vibrations are separately discussed in chapter 24.

δ = Static deflection of the spring in metres due to weight W newtons, and

x = Displacement given to the body by the external force, in metres.

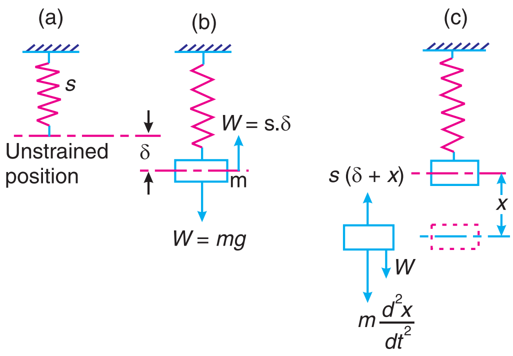


Fig. 23.2. Natural frequency of free longitudinal vibrations.

In the equilibrium position, as shown in Fig. 23.2 (*b*), the gravitational pull $W = m.g$, is balanced by a force of spring, such that $W = s.\delta$.

Since the mass is now displaced from its equilibrium position by a distance x, as shown in Fig. 23.2 (*c*), and is then released, therefore after time t,

Restoring force $= W - s(\delta + x) = W - s.\delta - s.x$

$= s.\delta - s.\delta - s.x = -s.x$ . . . ($\because W = s.\delta$) . . . ***(i)***

. . . (Taking upward force as negative)

and Accelerating force = Mass × Acceleration

$= m \times \dfrac{d^2x}{dt^2}$ . . . (Taking downward force as positive) . . . ***(ii)***

Equating equations ***(i)*** and ***(ii)***, the equation of motion of the body of mass m after time t is

$$m \times \frac{d^2x}{dt^2} = -s.x \quad \text{or} \quad m \times \frac{d^2x}{dt^2} + s.x = 0$$

$$\therefore \quad \frac{d^2x}{dt^2} + \frac{s}{m} \times x = 0 \qquad \text{. . . (iii)}$$

We know that the fundamental equation of simple harmonic motion is

$$\frac{d^2x}{dt^2} + \omega^2.x = 0 \qquad \text{. . . (iv)}$$

Comparing equations ***(iii)*** and ***(iv)***, we have

$$\omega = \sqrt{\frac{s}{m}}$$

$\therefore$ Time period, $t_p = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{m}{s}}$

and natural frequency, $$f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$$ . . . ($\because$ $m.g = s.\delta$)

Taking the value of g as 9.81 m/s^2 and δ in metres,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{9.81}{\delta}} = \frac{0.4985}{\sqrt{\delta}}\text{ Hz}$$

Note : The value of static deflection δ may be found out from the given conditions of the problem. For longitudinal vibrations, it may be obtained by the relation,

$$\frac{\text{Stress}}{\text{Strain}} = E \quad \text{or} \quad \frac{W}{A} \times \frac{l}{\delta} = E \quad \text{or} \quad \delta = \frac{W.l}{E.A}$$

where δ = Static deflection *i.e.* extension or compression of the constraint,

W = Load attached to the free end of constraint,

l = Length of the constraint,

E = Young's modulus for the constraint, and

A = Cross-sectional area of the constraint.

2. *Energy method*

We know that the kinetic energy is due to the motion of the body and the potential energy is with respect to a certain datum position which is equal to the amount of work required to move the body from the datum position. In the case of vibrations, the datum position is the mean or equilibrium position at which the potential energy of the body or the system is zero.

In the free vibrations, no energy is transferred to the system or from the system. Therefore the summation of kinetic energy and potential energy must be a constant quantity which is same at all the times. In other words,

$$\therefore \quad \frac{d}{dt}(K.E. + P.E.) = 0$$

We know that kinetic energy,

$$K.E. = \frac{1}{2} \times m\left(\frac{dx}{dt}\right)^2$$

This industrial compressor uses compressed air to power heavy-duty construction tools. Compressors are used for jobs, such as breaking up concrete or paving, drilling, pile driving, sand-blasting and tunnelling. A compressor works on the same principle as a pump. A piston moves backwards and forwards inside a hollow cylinder, which compresses the air and forces it into a hollow chamber. A pipe or hose connected to the chamber channels the compressed air to the tools.

Note : This picture is given as additional information and is not a direct example of the current chapter.

and potential energy,
$$P.E. = \left(\frac{0+s.x}{2}\right)x = \frac{1}{2}\times s.x^2$$
. . . ($\because$ *P.E.* = Mean force × Displacement)

$$\therefore \quad \frac{d}{dt}\left[\frac{1}{2}\times m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\times s.x^2\right] = 0$$

$$\frac{1}{2}\times m\times 2\times\frac{dx}{dt}\times\frac{d^2x}{dt^2} + \frac{1}{2}\times s\times 2x\times\frac{dx}{dt} = 0$$

or
$$m\times\frac{d^2x}{dt^2} + s.x = 0 \quad \text{or} \quad \frac{d^2x}{dt^2} + \frac{s}{m}\times x = 0$$
. . . (Same as before)

The time period and the natural frequency may be obtained as discussed in the previous method.

3. *Rayleigh's method*

In this method, the maximum kinetic energy at the mean position is equal to the maximum potential energy (or strain energy) at the extreme position. Assuming the motion executed by the vibration to be simple harmonic, then

$$x = X \sin \omega.t \qquad \text{. . . (i)}$$

where x = Displacement of the body from the mean position after time t seconds, and

X = Maximum displacement from mean position to extreme position.

Now, differentiating equation (*i*), we have

$$\frac{dx}{dt} = \omega\times X \cos \omega.t$$

Since at the mean position, t = 0, therefore maximum velocity at the mean position,

$$v = \frac{dx}{dt} = \omega.X$$

$\therefore$ Maximum kinetic energy at mean position

$$= \frac{1}{2}\times m.v^2 = \frac{1}{2}\times m.\omega^2.X^2 \qquad \text{. . . (ii)}$$

and maximum potential energy at the extreme position

$$= \left(\frac{0+s.X}{2}\right)X = \frac{1}{2}\times s.X^2 \qquad \text{. . . (iii)}$$

Equating equations (*ii*) and (*iii*),

$$\frac{1}{2}\times m.\omega^2.X^2 = \frac{1}{2}\times s.X^2 \quad \text{or} \quad \omega^2 = \frac{s}{m}, \text{ and } \omega = \sqrt{\frac{s}{m}}$$

$\therefore$ Time period,
$$t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{s}{m}}$$
. . . (Same as before)

and natural frequency, $f_n = \frac{1}{t_p} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{s}{m}}$ . . . (Same as before)

Note : In all the above expressions, ω is known as **natural circular frequency** and is generally denoted by ω_n.

23.6. Natural Frequency of Free Transverse Vibrations

Consider a shaft of negligible mass, whose one end is fixed and the other end carries a body of weight *W*, as shown in Fig. 23.3.

Let s = Stiffness of shaft,

δ = Static deflection due to weight of the body,

x = Displacement of body from mean position after time *t*.

m = Mass of body = W/g

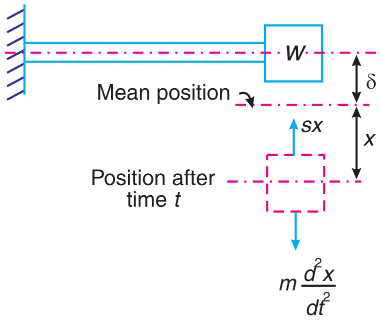


Fig. 23.3. Natural frequency of free transverse vibrations.

As discussed in the previous article,

Restoring force $= -s.x$ . . . (*i*)

and accelerating force $= m \times \frac{d^2x}{dt^2}$ . . . (*ii*)

Equating equations (*i*) and (*ii*), the equation of motion becomes

$$m \times \frac{d^2x}{dt^2} = -s.x \quad \text{or} \quad m \times \frac{d^2x}{dt^2} + s.x = 0$$

$$\therefore \quad \frac{d^2x}{dt^2} + \frac{s}{m} \times x = 0 \qquad \text{. . . (Same as before)}$$

Hence, the time period and the natural frequency of the transverse vibrations are same as that of longitudinal vibrations. Therefore

Time period, $t_p = 2\pi\sqrt{\frac{m}{s}}$

and natural frequency, $f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$

Note : The shape of the curve, into which the vibrating shaft deflects, is identical with the static deflection curve of a cantilever beam loaded at the end. It has been proved in the text book on Strength of Materials, that the static deflection of a cantilever beam loaded at the free end is

$$\delta = \frac{Wl^3}{3EI} \text{ (in metres)}$$

where W = Load at the free end, in newtons,

l = Length of the shaft or beam in metres,

E = Young's modulus for the material of the shaft or beam in N/m^2, and

I = Moment of inertia of the shaft or beam in m^4.

Example 23.1. *A cantilever shaft 50 mm diameter and 300 mm long has a disc of mass 100 kg at its free end. The Young's modulus for the shaft material is 200 GN/m^2. Determine the frequency of longitudinal and transverse vibrations of the shaft.*

Solution. Given : d = 50 mm = 0.05 m ; l = 300 mm = 0.03 m ; m = 100 kg ; E = 200 GN/m^2 = 200 ×10^9 N/m^2

We know that cross-sectional area of the shaft,

$$A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4}(0.05)^2 = 1.96 \times 10^{-3} \text{ m}^2$$

and moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.05)^4 = 0.3 \times 10^{-6} \text{ m}^4$$

Frequency of longitudinal vibration

We know that static deflection of the shaft,

$$\delta = \frac{W.l}{A.E} = \frac{100 \times 9.81 \times 0.3}{1.96 \times 10^{-3} \times 200 \times 10^9} = 0.751 \times 10^{-6} \text{ m}$$

$$\ldots (\because W = m.g)$$

∴ Frequency of longitudinal vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{0.751 \times 10^{-6}}} = 575 \text{ Hz} \quad \textbf{Ans.}$$

Frequency of transverse vibration

We know that static deflection of the shaft,

$$\delta = \frac{W.l^3}{3E.I} = \frac{100 \times 9.81 \times (0.3)^3}{3 \times 200 \times 10^9 \times 0.3 \times 10^{-6}} = 0.147 \times 10^{-3} \text{ m}$$

∴ Frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{0.147 \times 10^{-3}}} = 41 \text{ Hz} \quad \textbf{Ans.}$$

23.7. Effect of Inertia of the Constraint in Longitudinal and Transverse Vibrations

In deriving the expressions for natural frequency of longitudinal and transverse vibrations, we have neglected the inertia of the constraint *i.e.* shaft. We shall now discuss the effect of the inertia of the constraint, as below :

1. *Longitudinal vibration*

Consider the constraint whose one end is fixed and other end is free as shown in Fig. 23.4.

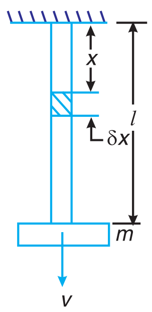


Fig. 23.4. Effect of inertia of the constraint in longitudinal vibrations.

Let m_1 = Mass of the constraint per unit length,

l = Length of the constraint,

m_C = Total mass of the constraint = $m_1 . l$, and

v = Longitudinal velocity of the free end.

Consider a small element of the constraint at a distance x from the fixed end and of length δx.

∴ Velocity of the small element

$$= \frac{x}{l} \times v$$

and kinetic energy possessed by the element

$$= \frac{1}{2} \times \text{Mass (velocity)}^2$$

$$= \frac{1}{2} \times m_1.\delta x \left(\frac{x}{l} \times v\right)^2 = \frac{m_1.v^2\, x^2}{2l^2} \times \delta x$$

∴ Total kinetic energy possessed by the constraint,

$$= \int_0^l \frac{m_1.v^2 x^2}{2l^2} \times dx = \frac{m_1.v^2}{2l^2}\left[\frac{x^3}{3}\right]_0^l$$

$$= \frac{m_1.v^2}{2l^2} \times \frac{l^3}{3} = \frac{1}{2} \times m_1.v^2 \times \frac{l}{3} = \frac{1}{2}\left(\frac{m_1.l}{3}\right)v^2 = \frac{1}{2}\left(\frac{m_C}{3}\right)v^2 \quad \ldots (i)$$

. . . (Substituting $m_1 . l = m_C$)

If a mass of $\frac{m_C}{3}$ is placed at the free end and the constraint is assumed to be of negligible mass, then

Total kinetic energy possessed by the constraint

$$= \frac{1}{2}\left(\frac{m_C}{3}\right)v^2 \quad \ldots \text{[Same as equation } (i)] \ldots (ii)$$

Hence the two systems are dynamically same. Therefore, inertia of the constraint may be allowed for by adding one-third of its mass to the disc at the free end.

From the above discussion, we find that when the mass of the constraint m_C and the mass of the disc m at the end is given, then natural frequency of vibration,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{s}{m + \frac{m_C}{3}}}$$

2. *Transverse vibration*

Consider a constraint whose one end is fixed and the other end is free as shown in Fig. 23.5.

Let m_1 = Mass of constraint per unit length,

l = Length of the constraint,

m_C = Total mass of the constraint = $m_1.l$, and

v = Transverse velocity of the free end.

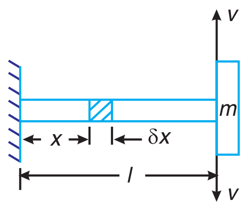


Fig. 23.5. Effect of inertia of the constraint in transverse vibrations.

Consider a small element of the constraint at a distance x from the fixed end and of length δx. The velocity of this element is given by $\left[\dfrac{3l.x^2 - x^3}{2l^3} \times v\right]$.

$\therefore$ Kinetic energy of the element

$$= \frac{1}{2} \times m_1.\delta x \left(\frac{3l.x^2 - x^3}{2l^3} \times v\right)^2$$

and total kinetic energy of the constraint,

$$= \int_0^l \frac{1}{2} \times m_1 \left(\frac{3l.x^2 - x^3}{2l^3} \times v\right)^2 dx = \frac{m_1.v^2}{8l^6} \int_0^l (9l^2.x^4 - 6l.x^5 + x^6)\, dx$$

$$= \frac{m_1.v^2}{8l^6} \left[\frac{9l^2.x^5}{5} - \frac{6l.x^6}{6} + \frac{x^7}{7}\right]_0^l$$

$$= \frac{m_1.v^2}{8l^6} \left[\frac{9l^7}{5} - \frac{6l^7}{6} + \frac{l^7}{7}\right] = \frac{m_1.v^2}{8l^6} \left(\frac{33l^7}{35}\right)$$

$$= \frac{33}{280} \times m_1.l.v^2 = \frac{1}{2}\left(\frac{33}{140} \times m_1.l\right)v^2 = \frac{1}{2}\left(\frac{33}{140} \times m_C\right)v^2 \qquad \ldots (i)$$

. . . (Substituting $m_1.l = m_C$)

If a mass of $\dfrac{33 m_C}{140}$ is placed at the free end and the constraint is assumed to be of negligible mass, then

Total kinetic energy possessed by the constraint

$$= \frac{1}{2}\left(\frac{33 m_C}{140}\right)v^2 \qquad \ldots \text{[Same as equation } (i)\text{]}$$

Hence the two systems are dynamically same. Therefore the inertia of the constraint may be allowed for by adding $\dfrac{33}{140}$ of its mass to the disc at the free end.

From the above discussion, we find that when the mass of the constraint m_C and the mass of the disc m at the free end is given, then natural frequency of vibration,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{s}{m + \dfrac{33 m_C}{140}}}$$

Notes : 1. If both the ends of the constraint are fixed, and the disc is situated in the middle of it, then proceeding in the similar way as discussed above, we may prove that the inertia of the constraint may be allowed for by adding $\dfrac{13}{35}$ of its mass to the disc.

2. If the constraint is like a simply supported beam, then $\dfrac{17}{35}$ of its mass may be added to the mass of the disc.

23.8. Natural Frequency of Free Transverse Vibrations Due to a Point Load Acting Over a Simply Supported Shaft

Consider a shaft *AB* of length *l*, carrying a point load *W* at *C* which is at a distance of l_1 from *A* and l_2 from *B*, as shown in Fig. 23.6. A little consideration will show that when the shaft is deflected and suddenly released, it will make transverse vibrations. The deflection of the shaft is proportional to the load *W* and if the beam is deflected beyond the static equilibrium position then the load will vibrate with simple harmonic motion (as by a helical spring). If δ is the static deflection due to load *W*, then the natural frequency of the free transverse vibration is

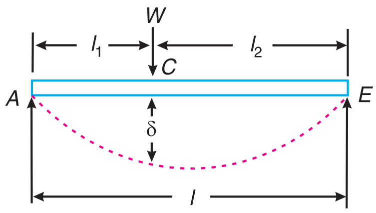
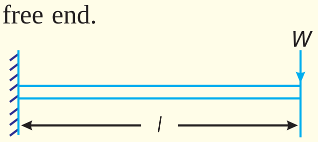
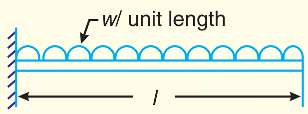
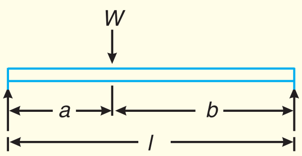
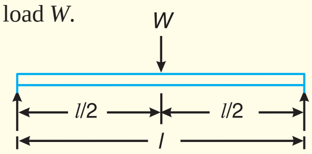


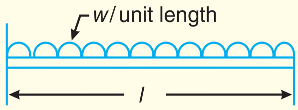
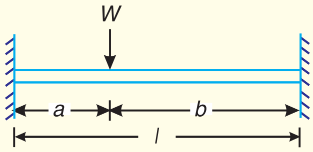
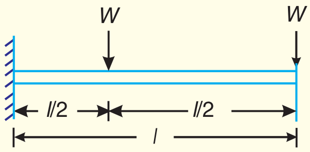
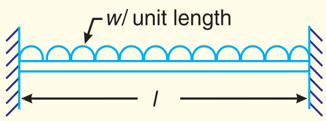
Fig. 23.6. Simply supported beam with a point load.

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{0.4985}{\sqrt{\delta}} \text{ Hz} \qquad \ldots (\text{Substituting, } g = 9.81 \text{ m/s}^2)$$

Some of the values of the static deflection for the various types of beams and under various load conditions are given in the following table.

Table 23.1. Values of static deflection (δ) for the various types of beams and under various load conditions.

S.No.	*Type of beam*	*Deflection (δ)*
1.	Cantilever beam with a point load *W* at the free end. 	$\delta = \frac{Wl^3}{3EI}$ (at the free end)
2.	Cantilever beam with a uniformly distributed load of *w* per unit length. 	$\delta = \frac{wl^4}{8EI}$ (at the free end)
3.	Simply supported beam with an eccentric point load *W*. 	$\delta = \frac{Wa^2b^2}{3EIl}$ (at the point load)
4.	Simply supported beam with a central point load *W*. 	$\delta = \frac{Wl^3}{48EI}$ (at the centre)

S.No.	Type of beam	Deflection (δ)
5.	Simply supported beam with a uniformly distributed load of *w* per unit length. 	$\delta = \frac{5}{384} \times \frac{wl^4}{EI}$ (at the centre)
6.	Fixed beam with an eccentric point load *W*. 	$\delta = \frac{Wa^3b^3}{3E\,I\,l}$ (at the point load)
7.	Fixed beam with a central point load *W*. 	$\delta = \frac{Wl^3}{192EI}$ (at the centre)
8.	Fixed beam with a uniformly distributed load of *w* per unit length. 	$\delta = \frac{wl^4}{384EI}$ (at the centre)

Example 23.2. *A shaft of length 0.75 m, supported freely at the ends, is carrying a body of mass 90 kg at 0.25 m from one end. Find the natural frequency of transverse vibration. Assume E = 200 GN/m² and shaft diameter = 50 mm.*

Solution. Given : $l = 0.75$ m ; $m = 90$ kg ; $a = AC = 0.25$ m ; $E = 200$ GN/m^2 = 200×10^9 N/m^2; $d = 50$ mm = 0.05 m

The shaft is shown in Fig. 23.7.

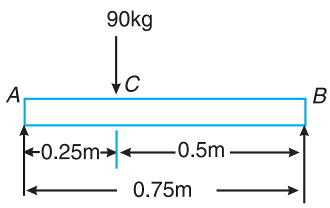


Fig. 23.7

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.05)^4 \text{ m}^4$$

$$= 0.307 \times 10^{-6} \text{ m}^4$$

and static deflection at the load point (*i.e.* at point *C*),

$$\delta = \frac{Wa^2b^2}{3E\,I\,l} = \frac{90 \times 9.81 (0.25)^2 (0.5)^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 0.75} = 0.1 \times 10^{-3} \text{ m}$$

. . . ($\because$ $b = BC = 0.5$ m)

We know that natural frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{0.1 \times 10^{-3}}} = 49.85 \text{ Hz} \quad \textbf{Ans.}$$

Example 23.3. *A flywheel is mounted on a vertical shaft as shown in Fig. 23.8. The both ends of the shaft are fixed and its diameter is 50 mm. The flywheel has a mass of 500 kg. Find the natural frequencies of longitudinal and transverse vibrations. Take E = 200 GN/m².*

Solution. Given : d = 50 mm = 0.05 m ; m = 500 kg ; E = 200 GN/m² = 200 × 10⁹ N/m²

We know that cross-sectional area of shaft,

$$A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4}(0.05)^2 = 1.96 \times 10^{-3} \text{ m}^2$$

and moment of inertia of shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.05)^4 = 0.307 \times 10^{-6} \text{ m}^4$$

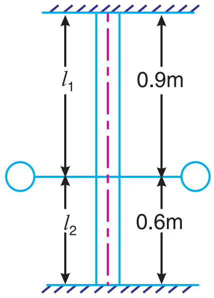


Fig. 23.8

Natural frequency of longitudinal vibration

Let m_1 = Mass of flywheel carried by the length l_1.

∴ $m - m_1$ = Mass of flywheel carried by length l_2.

We know that extension of length l_1

$$= \frac{W_1 . l_1}{A.E} = \frac{m_1 . g . l_1}{A.E} \quad \ldots (i)$$

Similarly, compression of length l_2

$$= \frac{(W - W_1)\, l_2}{A.E} = \frac{(m - m_1)\, g . l_2}{A.E} \quad \ldots (ii)$$

Since extension of length l_1 must be equal to compression of length l_2, therefore equating equations (*i*) and (*ii*),

$$m_1 . l_1 = (m - m_1)\, l_2$$

$$m_1 \times 0.9 = (500 - m_1)\, 0.6 = 300 - 0.6\, m_1 \quad \text{or} \quad m_1 = 200 \text{ kg}$$

∴ Extension of length l_1,

$$\delta = \frac{m_1 . g . l_1}{A.E} = \frac{200 \times 9.81 \times 0.9}{1.96 \times 10^{-3} \times 200 \times 10^9} = 4.5 \times 10^{-6} \text{ m}$$

We know that natural frequency of longitudinal vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{4.5 \times 10^{-6}}} = 235 \text{ Hz} \quad \textbf{Ans.}$$

Natural frequency of transverse vibration

We know that the static deflection for a shaft fixed at both ends and carrying a point load is given by

$$\delta = \frac{W a^3 b^3}{3 E I l^3} = \frac{500 \times 9.81 (0.9)^3 (0.6)^3}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} (1.5)^3} = 1.24 \times 10^{-3} \text{ m}$$

. . . (Substituting $W = m.g$; $a = l_1$, and $b = l_2$)

We know that natural frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{1.24 \times 10^{-3}}} = 14.24 \text{ Hz} \quad \textbf{Ans.}$$

23.9. Natural Frequency of Free Transverse Vibrations Due to Uniformly Distributed Load Acting Over a Simply Supported Shaft

Consider a shaft AB carrying a uniformly distributed load of w per unit length as shown in Fig. 23.9.

Let y_1 = Static deflection at the middle of the shaft,

a_1 = Amplitude of vibration at the middle of the shaft, and

w_1 = Uniformly distributed load per unit static deflection at the middle of the shaft = w/y_1.

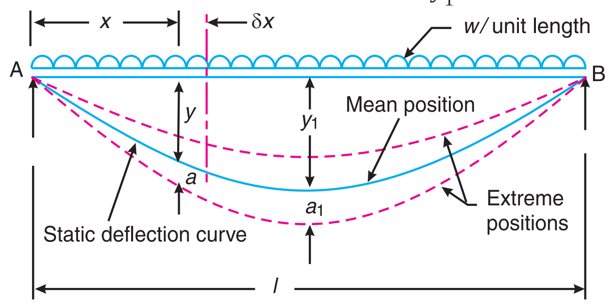


Fig. 23.9. Simply supported shaft carrying a uniformly distributed load.

Now, consider a small section of the shaft at a distance x from A and length δx.

Let y = Static deflection at a distance x from A, and

a = Amplitude of its vibration.

∴ Work done on this small section

$$= \frac{1}{2} \times w_1.a_1.\delta x \times a = \frac{1}{2} \times \frac{w}{y_1} \times a_1.\delta x \times a = \frac{1}{2} \times w \times \frac{a_1}{y_1} \times a \times \delta x$$

Since the maximum potential energy at the extreme position is equal to the amount of work done to move the beam from the mean position to one of its extreme positions, therefore

Maximum potential energy at the extreme position

$$= \int_0^l \frac{1}{2} \times w \times \frac{a_1}{y_1} \times a.dx \qquad \ldots (i)$$

Assuming that the shape of the curve of a vibrating shaft is similar to the static deflection curve of a beam, therefore

$$\frac{a_1}{y_1} = \frac{a}{y} = \text{Constant, } C \quad \text{or} \quad \frac{a_1}{y_1} = C \text{ and } a = y.C$$

Substituting these values in equation (*i*), we have maximum potential energy at the extreme position

$$= \int_0^l \frac{1}{2} \times w \times C \times y.C.dx = \frac{1}{2} \times w.C^2 \int_0^l y.dx \qquad \ldots (ii)$$

Since the maximum velocity at the mean position is $\omega . a_1$, where ω is the circular frequency of vibration, therefore

Maximum kinetic energy at the mean position

$$= \int_0^l \frac{1}{2} \times \frac{w.dx}{g} (\omega . a)^2 = \frac{w}{2g} \times \omega^2 \times C^2 \int_0^l y^2 .dx \qquad \ldots (iii)$$

. . .(Substituting $a = y.C$)

We know that the maximum potential energy at the extreme position is equal to the maximum kinetic energy at the mean position, therefore equating equations **(*ii*)** and **(*iii*),**

$$\frac{1}{2} \times w \times C^2 \int_0^l y.dx = \frac{w}{2g} \times \omega^2 \times C^2 \int_0^l y^2 .dx$$

$$\therefore \qquad \omega^2 = \frac{g \int_0^l y.dx}{\int_0^l y^2 .dx} \quad \text{or} \quad \omega = \sqrt{\frac{g \int_0^l y.dx}{\int_0^l y^2 .dx}} \qquad \ldots (iv)$$

When the shaft is a simply supported, then the static deflection at a distance x from A is

$$^* \; y = \frac{w}{24\,EI} (x^4 - 2l\,x^3 + l^3 x) \qquad \ldots (v)$$

where w = Uniformly distributed load unit length,

E = Young's modulus for the material of the shaft, and

I = Moment of inertia of the shaft.

---

* It has been proved in books on 'Strength of Materials' that maximum bending moment at a distance x from A is

$$(B.M.)_{max} = EI \frac{d^2 y}{dx^2} = \frac{wx^2}{2} - \frac{wl\,x}{2}$$

Integrating this expression,

$$EI.\frac{dy}{dx} = \frac{wx^3}{2 \times 3} - \frac{wl.x^2}{2 \times 2} + C_1$$

On further integrating,

$$E.I.y = \frac{wx^4}{2 \times 3 \times 4} - \frac{wl.x^3}{2 \times 2 \times 3} + C_1 x + C_2$$

$$= \frac{wx^4}{24} - \frac{wlx^3}{12} + C_1 x + C_2$$

where C_1 and C_2 are the constants of integration and may be determined from the given conditions of the problem. Here

when $x = 0, y = 0$; $\therefore$ $C_2 = 0$

and when $x = l, y = 0$; $\therefore$ $C_1 = \frac{wl^3}{24}$

Substituting the value of C_1, we get

$$y = \frac{w}{24\,EI} (x^4 - 2l\,x^3 + l^3 x)$$


A railway bridge.

Now integrating the above equation *(v)* within the limits from 0 to l,

$$\int_0^l y\,dx = \frac{w}{24\,EI}\int_0^l (x^4 - 2\,lx^3 + l^3 x)\,dx = \frac{w}{24\,EI}\left[\frac{x^5}{5} - \frac{2\,lx^4}{4} + \frac{l^3 x^2}{2}\right]_0^l$$

$$= \frac{w}{24\,EI}\left[\frac{l^5}{5} - \frac{2\,l^5}{4} + \frac{l^5}{2}\right] = \frac{w}{24\,EI} \times \frac{l^5}{5} = \frac{w.l^5}{120\,E.I} \qquad \ldots (vi)$$

Now
$$\int_0^l y^2 dx = \int_0^l \left[\frac{w}{24\,EI}(x^4 - 2l\,x^3 + l^3 x)\right]^2 dx$$

$$= \left(\frac{w}{24\,EI}\right)^2 \int_0^l (x^8 + 4\,l^2 x^6 + l^6 x^2 - 4l\,x^7 - 4\,l^4 x^4 + 2\,l^3 x^5)\,dx$$

$$= \frac{w^2}{576\,E^2\,I^2} \cdot \left[\frac{x^9}{9} + \frac{4\,l^2 x^7}{7} + \frac{l^6 x^3}{3} - \frac{4\,lx^8}{8} - \frac{4\,l^4 x^5}{5} + \frac{2\,l^3 x^6}{6}\right]_0^l$$

$$= \frac{w^2}{576\,E^2\,I^2}\left[\frac{l^9}{9} + \frac{4\,l^9}{7} + \frac{l^9}{3} - \frac{4\,l^9}{8} - \frac{4\,l^9}{5} + \frac{2\,l^9}{6}\right]$$

$$= \frac{w^2}{576\,E^2\,I^2} \times \frac{31\,l^9}{630} \qquad \ldots (vii)$$

Substituting the value in equation *(iv)* from equations *(vi)* and *(vii)*, we get circular frequency due to uniformly distributed load,

$$\omega = \sqrt{g\left(\frac{wl^5}{120\,EI} \times \frac{576\,E^2 I^2 \times 630}{w^2 \times 31\,l^9}\right)}$$

$$= \sqrt{\frac{24\,EI}{wl^4} \times \frac{630}{155} g} = \pi^2 \sqrt{\frac{EI\,g}{wl^4}} \qquad \ldots (viii)$$

∴ Natural frequency due to uniformly distributed load,

$$f_n = \frac{\omega}{2\pi} = \frac{\pi^2}{2\pi} \sqrt{\frac{EI\,g}{wl^4}} = \frac{\pi}{2} \sqrt{\frac{EIg}{wl^4}} \qquad \ldots (ix)$$

We know that the static deflection of a simply supported shaft due to uniformly distributed load of w per unit length, is

$$\delta_S = \frac{5\,wl^4}{384\,EI} \quad \text{or} \quad \frac{EI}{wl^4} = \frac{5}{384\,\delta_S}$$

Equation **(*ix*)** may be written as

$$f_n = \frac{\pi}{2} \sqrt{\frac{5\,g}{384\,\delta_S}} = \frac{0.5615}{\sqrt{\delta_S}} \text{ Hz} \qquad \ldots \text{(Substituting, } g = 9.81 \text{ m/s}^2\text{)}$$

23.10. Natural Frequency of Free Transverse Vibrations of a Shaft Fixed at Both Ends Carrying a Uniformly Distributed Load

Consider a shaft AB fixed at both ends and carrying a uniformly distributed load of w per unit length as shown in Fig. 23.10.

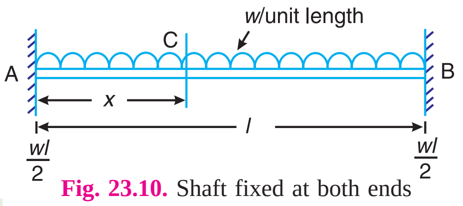


Fig. 23.10. Shaft fixed at both ends carrying a uniformly distributed load.

We know that the static deflection at a distance x from A is given by

$$*\; y = \frac{w}{24\,EI}(x^4 + l^2x^2 - 2lx^3) \qquad \ldots (i)$$

* It has been proved in books on 'Strength of Materials' that the bending moment at a distance x from A is

$$M = EI\frac{d^2y}{dx^2} = \frac{wl^2}{12} + \frac{wx^2}{2} - \frac{wlx}{2}$$

Integrating this equation,

$$EI\frac{dy}{dx} = \frac{wl^2}{12}x + \frac{wx^3}{2 \times 3} - \frac{wlx^2}{2 \times 2} + C_1$$

where C_1 is the constant of integration. We know that when $x = 0, \frac{dy}{dx} = 0$. Therefore $C_1 = 0$.

or $$EI\frac{dy}{dx} = \frac{wl^2}{12}x + \frac{wx^3}{6} - \frac{wlx^2}{4}$$

Integrating the above equation,

$$EI.y = \frac{wl^2x^2}{12 \times 2} + \frac{wx^4}{6 \times 4} - \frac{wl}{4} \times \frac{x^3}{3} + C = \frac{wl^2x^2}{24} + \frac{wx^4}{24} - \frac{wlx^3}{12} + C_2$$

where C_2 is the constant of integration. We know that when $x = 0$, $y = 0$. Therefore $C_2 = 0$.

or $$EI.y = \frac{w}{24}(l^2x^2 + x^4 - 2lx^3)$$

or $$y = \frac{w}{24\,EI}(x^4 + l^2x^2 - 2\,lx^3)$$

Integrating the above equation within limits from 0 to l,

$$\int_0^l y\,dx = \frac{w}{24\,EI}\int_0^l (x^4 + l^2x^2 - 2l\,x^3)\,dx$$

$$= \frac{w}{24\,EI}\left[\frac{x^5}{5} + \frac{l^2x^3}{3} - \frac{2l\,x^4}{4}\right]_0^l = \frac{w}{24\,EI}\left[\frac{l^5}{5} + \frac{l^5}{3} - \frac{2l^5}{4}\right]$$

$$= \frac{w}{24\,EI} \times \frac{l^5}{30} = \frac{w\,l^5}{720\,EI}$$

Now integrating y^2 within the limits from 0 to l,

$$\int_0^l y^2\,dx = \left(\frac{w}{24\,EI}\right)^2 \int_0^l (x^4 + l^2x^2 - 2l\,x^3)^2\,dx$$

$$= \left(\frac{w}{24\,EI}\right)^2 \int_0^l (x^8 + l^4x^4 + 4l^2x^6 + 2l^2x^6 - 4l\,x^7 - 2l^3x^5)\,dx$$

$$= \left(\frac{w}{24\,EI}\right)^2 \int_0^l (x^8 + l^4x^4 + 6l^2x^6 + 4l\,x^7 - 2l^3x^5)\,dx$$

$$= \left(\frac{w}{24\,EI}\right)^2 \left[\frac{x^9}{9} + \frac{l^4x^5}{5} + \frac{6l^2x^7}{7} - \frac{4l\,x^8}{8} - \frac{2l^3x^6}{6}\right]_0^l$$

$$= \left(\frac{w}{24\,EI}\right)^2 \left[\frac{l^9}{9} + \frac{l^9}{5} + \frac{6l^9}{7} - \frac{4l^9}{8} - \frac{2l^9}{6}\right] = \left(\frac{w}{24\,EI}\right)^2 \frac{l^9}{630}$$

We know that

$$\omega^2 = \frac{g\int_0^l y\,dx}{\int_0^l y^2\,dx} = g \times \frac{wl^5}{720\,EI} \times \frac{(24\,EI)^2 \times 630}{w^2l^9} = \frac{504\,EIg}{wl^4}$$

$\therefore$ $$\omega = \sqrt{\frac{504\,EIg}{wl^4}}$$

and natural frequency,

$$f_n = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{504\,EIg}{wl^4}} = 3.573\sqrt{\frac{E\,I\,g}{w\,l^4}}$$

Since the static deflection of a shaft fixed at both ends and carrying a uniformly distributed load is

$$\delta_S = \frac{wl^4}{384\,EI} \quad \text{or} \quad \frac{E\,I}{wl^4} = \frac{1}{384\,\delta_S}$$

$\therefore$ $$f_n = 3.573\sqrt{\frac{g}{384\,\delta_S}} = \frac{0.571}{\sqrt{\delta_S}}\ \text{Hz}$$ . . . (Substituting, g = 9.81 m/s^2)

23.11. Natural Frequency of Free Transverse Vibrations For a Shaft Subjected to a Number of Point Loads

Consider a shaft *AB* of negligible mass loaded with point loads W_1, W_2, W_3 and W_4 etc. in newtons, as shown in Fig. 23.11. Let m_1, m_2, m_3 and m_4 etc. be the corresponding masses in kg. The natural frequency of such a shaft may be found out by the following two methods :

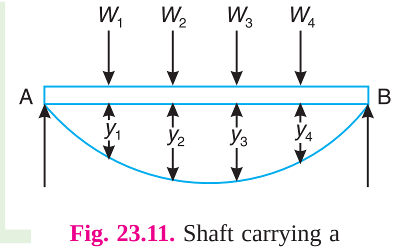


Fig. 23.11. Shaft carrying a number of point loads.

1. *Energy (or Rayleigh's) method*

Let y_1, y_2, y_3, y_4 etc. be total deflection under loads W_1, W_2, W_3 and W_4 etc. as shown in Fig. 23.11.

We know that maximum potential energy

$$= \frac{1}{2} \times m_1.g.y_1 + \frac{1}{2} \times m_2.g.y_2 + \frac{1}{2} m_3.g.y_3 + \frac{1}{2} \times m_4.g.y_4 + ...$$

$$= \frac{1}{2} \Sigma m.g.y$$

and maximum kinetic energy

$$= \frac{1}{2} \times m_1 (\omega.y_1)^2 + \frac{1}{2} \times m_2 (\omega.y_2)^2 + \frac{1}{2} \times m_3 (\omega.y_3)^2 + \frac{1}{2} \times m_4 (\omega.y_4)^2 + ...$$

$$= \frac{1}{2} \times \omega^2 \left[m_1 (y_1)^2 + m_2 (y_2)^2 + m_3 (y_3)^2 + m_4 (y_4)^2 + ... \right]$$

$$= \frac{1}{2} \times \omega^2 \Sigma m.y^2 \qquad \text{. . . (where } \omega = \text{Circular frequency of vibration)}$$

Equating the maximum kinetic energy to the maximum potential energy, we have

$$\frac{1}{2} \times \omega^2 \, \Sigma m.y^2 = \frac{1}{2} \Sigma m.g.y$$

$$\therefore \qquad \omega^2 = \frac{\Sigma m.g.y}{\Sigma m.y^2} = \frac{g \, \Sigma m.y}{\Sigma m.y^2} \qquad \text{or} \qquad \omega = \sqrt{\frac{g \, \Sigma m.y}{\Sigma m.y^2}}$$

$\therefore$ Natural frequency of transverse vibration,

$$f_n = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{g \, \Sigma m.y}{\Sigma m.y^2}}$$

2. *Dunkerley's method*

The natural frequency of transverse vibration for a shaft carrying a number of point loads and uniformly distributed load is obtained from Dunkerley's empirical formula. According to this

$$\frac{1}{(f_n)^2} = \frac{1}{(f_{n1})^2} + \frac{1}{(f_{n2})^2} + \frac{1}{(f_{n3})^2} + ... + \frac{1}{(f_{ns})^2}$$

where f_n = Natural frequency of transverse vibration of the shaft carrying point loads and uniformly distributed load.

f_{n1}, f_{n2}, f_{n3}, etc. = Natural frequency of transverse vibration of each point load.

f_{ns} = Natural frequency of transverse vibration of the uniformly distributed load (or due to the mass of the shaft).

Now, consider a shaft *AB* loaded as shown in Fig. 23.12.

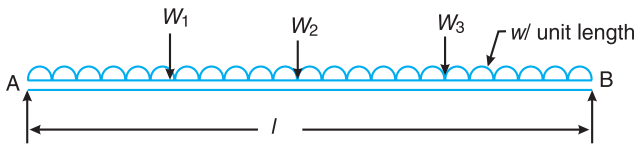


Fig. 23.12. Shaft carrying a number of point loads and a uniformly distributed load.

Let $\delta_1, \delta_2, \delta_3$, etc. = Static deflection due to the load W_1, W_2, W_3 etc. when considered separately.

δ_S = Static deflection due to the uniformly distributed load or due to the mass of the shaft.

We know that natural frequency of transverse vibration due to load W_1,

$$f_{n_1} = \frac{0.4985}{\sqrt{\delta_1}} \text{ Hz}$$

Similarly, natural frequency of transverse vibration due to load W_2,

$$f_{n_2} = \frac{0.4985}{\sqrt{\delta_2}} \text{ Hz}$$

and, natural frequency of transverse vibration due to load W_3,

$$f_{n_3} = \frac{0.4985}{\sqrt{\delta_3}} \text{ Hz}$$

Also natural frequency of transverse vibration due to uniformly distributed load or weight of the shaft,

$$f_{ns} = \frac{0.5615}{\sqrt{\delta_S}} \text{ Hz}$$

Suspension spring of an automobile.

Note : This picture is given as additional information and is not a direct example of the current chapter.

Therefore, according to Dunkerley's empirical formula, the natural frequency of the whole system,

$$\frac{1}{(f_n)^2} = \frac{1}{(f_{n1})^2} + \frac{1}{(f_{n2})^2} + \frac{1}{(f_{n3})^3} + ... + \frac{1}{(f_{ns})^2}$$

$$= \frac{\delta_1}{(0.4985)^2} + \frac{\delta_2}{(0.4985)^2} + \frac{\delta_3}{(0.4985)^2} + ... + \frac{\delta_S}{(0.5615)^2}$$

$$= \frac{1}{(0.4985)^2}\left[\delta_1 + \delta_2 + \delta_3 + ... + \frac{\delta_S}{1.27}\right]$$

or
$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \delta_3 + ... + \frac{\delta_S}{1.27}}} \text{ Hz}$$

Notes : 1. When there is no uniformly distributed load or mass of the shaft is negligible, then $\delta_S = 0$.

$$\therefore \quad f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \delta_3 + ...}} \text{ Hz}$$

2. The value of δ_1, δ_2, δ_3 etc. for a simply supported shaft may be obtained from the relation

$$\delta = \frac{Wa^2b^2}{3EIl}$$

where δ = Static deflection due to load W,

a and b = Distances of the load from the ends,

E = Young's modulus for the material of the shaft,

I = Moment of inertia of the shaft, and

l = Total length of the shaft.

Example 23.4. *A shaft 50 mm diameter and 3 metres long is simply supported at the ends and carries three loads of 1000 N, 1500 N and 750 N at 1 m, 2 m and 2.5 m from the left support. The Young's modulus for shaft material is 200 GN/m². Find the frequency of transverse vibration.*

Solution. Given : d = 50 mm = 0.05 m ; l = 3 m, W_1 = 1000 N ; W_2 = 1500 N ; W_3 = 750 N; E = 200 GN/m² = 200 × 10^9 N/m²

The shaft carrying the loads is shown in Fig. 23.13

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.05)^4 = 0.307 \times 10^{-6} \text{ m}^4$$

and the static deflection due to a point load W,

$$\delta = \frac{Wa^2b^2}{3EIl}$$

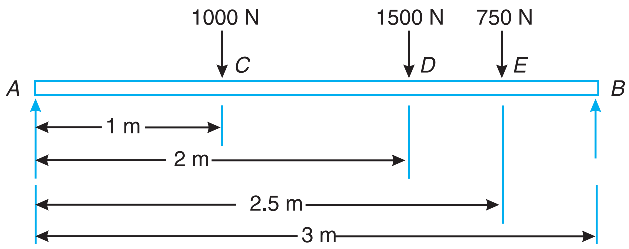


Fig. 23.13

$\therefore$ Static deflection due to a load of 1000 N,

$$\delta_1 = \frac{1000 \times 1^2 \times 2^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 3} = 7.24 \times 10^{-3} \text{ m}$$

. . . (Here a = 1 m, and b = 2 m)

Similarly, static deflection due to a load of 1500 N,

$$\delta_2 = \frac{1500 \times 2^2 \times 1^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 3} = 10.86 \times 10^{-3} \text{ m}$$

. . . (Here a = 2 m, and b = 1 m)

and static deflection due to a load of 750 N,

$$\delta_3 = \frac{750\,(2.5)^2\,(0.5)^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 3} = 2.12 \times 10^{-3} \text{ m}$$

. . . (Here a = 2.5 m, and b = 0.5 m)

We know that frequency of transverse vibration,

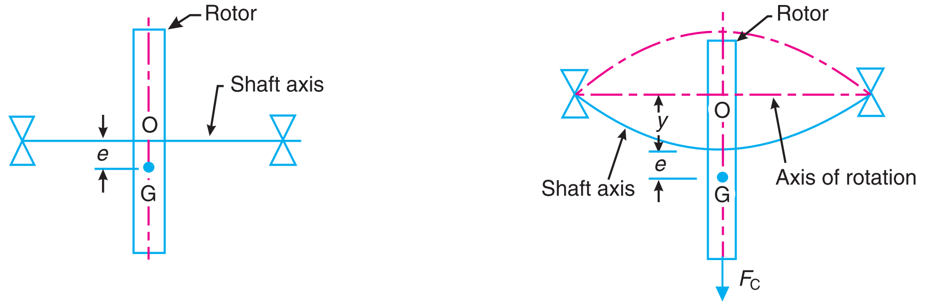
$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \delta_3}} = \frac{0.4985}{\sqrt{7.24 \times 10^{-3} + 10.86 \times 10^{-3} + 2.12 \times 10^{-3}}}$$

$$= \frac{0.4985}{0.1422} = 3.5 \text{ Hz}$$ **Ans.**

23.12. Critical or Whirling Speed of a Shaft

In actual practice, a rotating shaft carries different mountings and accessories in the form of gears, pulleys, etc. When the gears or pulleys are put on the shaft, the centre of gravity of the pulley or gear does not coincide with the centre line of the bearings or with the axis of the shaft, when the shaft is stationary. This means that the centre of gravity of the pulley or gear is at a certain distance from the axis of rotation and due to this, the shaft is subjected to centrifugal force. This force will bent the shaft which will further increase the distance of centre of gravity of the pulley or gear from the axis of rotation. This correspondingly increases the value of centrifugal force, which further increases the distance of centre of gravity from the axis of rotation. This effect is cumulative and ultimately the shaft fails. The bending of shaft not only depends upon the value of eccentricity (distance between centre of gravity of the pulley and the axis of rotation) but also depends upon the speed at which the shaft rotates.

The speed at which the shaft runs so that the additional deflection of the shaft from the axis of rotation becomes infinite, is known as *critical* or *whirling speed*.



(*a*) When shaft is stationary. (*b*) When shaft is rotating.

Fig. 23.14. Critical or whirling speed of a shaft.

Consider a shaft of negligible mass carrying a rotor, as shown in Fig.23.14 (*a*). The point O is on the shaft axis and G is the centre of gravity of the rotor. When the shaft is stationary, the centre line of the bearing and the axis of the shaft coincides. Fig. 23.14 (*b*) shows the shaft when rotating about the axis of rotation at a uniform speed of ω rad/s.

Let m = Mass of the rotor,

e = Initial distance of centre of gravity of the rotor from the centre line of the bearing or shaft axis, when the shaft is stationary,

y = Additional deflection of centre of gravity of the rotor when the shaft starts rotating at ω rad/s, and

s = Stiffness of the shaft *i.e.* the load required per unit deflection of the shaft.

Since the shaft is rotating at ω rad/s, therefore centrifugal force acting radially outwards through G causing the shaft to deflect is given by

$$F_C = m.\omega^2 (y + e)$$

The shaft behaves like a spring. Therefore the force resisting the deflection y,

$$= s.y$$

For the equilibrium position,

$$m.\omega^2 (y + e) = s.y$$

or $$m.\omega^2.y + m.\omega^2.e = s.y \quad \text{or} \quad y(s - m.\omega^2) = m.\omega^2.e$$

$$\therefore \quad y = \frac{m.\omega^2.e}{s - m.\omega^2} = \frac{\omega^2.e}{s/m - \omega^2} \quad \ldots (i)$$

We know that circular frequency,

$$\omega_n = \sqrt{\frac{s}{m}} \quad \text{or} \quad y = \frac{\omega^2.e}{(\omega_n)^2 - \omega^2} \quad \ldots [\text{From equation } (i)]$$

A little consideration will show that when $\omega > \omega_n$, the value of y will be negative and the shaft deflects is the opposite direction as shown dotted in Fig 23.14 (*b*).

In order to have the value of y always positive, both ***plus*** and ***minus*** signs are taken.

$$\therefore \quad y = \pm \frac{\omega^2 e}{(\omega_n)^2 - \omega^2} = \frac{\pm e}{\left(\frac{\omega_n}{\omega}\right)^2 - 1} = \frac{\pm e}{\left(\frac{\omega_c}{\omega}\right)^2 - 1}$$

... (Substituting $\omega_n = \omega_c$)

We see from the above expression that when $\omega_n = \omega_c$, the value of y becomes infinite. Therefore ω_c is the **critical or whirling speed.**

$\therefore$ Critical or whirling speed,

$$\omega_c = \omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{g}{\delta}} \text{ Hz} \quad \ldots \left(\because \delta = \frac{m.g}{s}\right)$$

If N_c is the critical or whirling speed in r.p.s., then

$$2\pi N_c = \sqrt{\frac{g}{\delta}} \quad \text{or} \quad N_c = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{0.4985}{\sqrt{\delta}} \text{ r.p.s.}$$

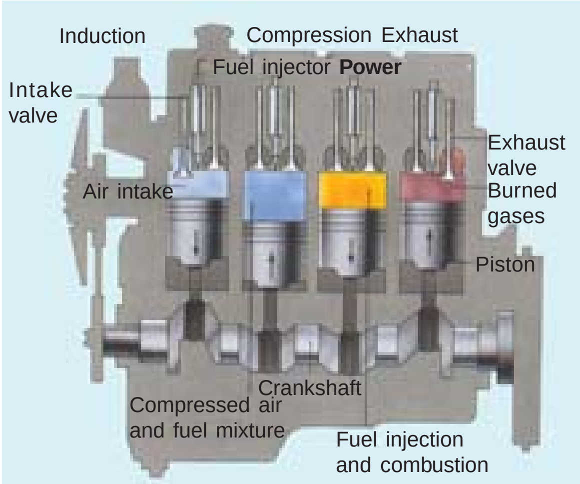
where δ = Static deflection of the shaft in metres.

Hence the **critical or whirling speed is the same as the natural frequency of transverse vibration but its unit will be revolutions per second.**

Notes : 1. When the centre of gravity of the rotor lies between the centre line of the shaft and the centre line of the bearing, *e* is taken negative. On the other hand, if the centre of gravity of the rotor does not lie between the centre line of the shaft and the centre line of the bearing (as in the above article) the value of *e* is taken positive.

2. To determine the critical speed of a shaft which may be subjected to point loads, uniformly distributed load or combination of both, find the frequency of transverse vibration which is equal to critical speed of a shaft in r.p.s. The Dunkerley's method may be used for calculating the frequency.

3. A shaft supported is short bearings (or ball bearings) is assumed to be a simply supported shaft while the shaft supported in long bearings (or journal bearings) is assumed to have both ends fixed.



Diesel engines have several advantages over petrol engines. They do not need an electrical ignition system; they use cheaper fuel; and they do not need a carburettor. Diesel engines also have a greater ability to convert the stored energy in the fuel into mechanical energy, or work.

Note : This picture is given as additional information and is not a direct example of the current chapter.

Example 23.5. *Calculate the whirling speed of a shaft 20 mm diameter and 0.6 m long carrying a mass of 1 kg at its mid-point. The density of the shaft material is 40 Mg/m³, and Young's modulus is 200 GN/m². Assume the shaft to be freely supported.*

Solution. Given : d = 20 mm = 0.02 m ; l = 0.6 m ; m_1 = 1 kg ; ρ = 40 Mg/m³ = 40×10^6 g/m³ = 40×10^3 kg/m³ ; E = 200 GN/m² = 200×10^9 N/m²

The shaft is shown in Fig. 23.15.

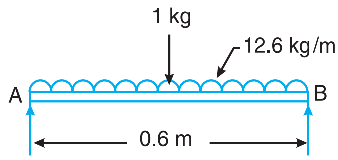


Fig. 23.15

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.02)^4 \text{ m}^4$$

$$= 7.855 \times 10^{-9} \text{ m}^4$$

Since the density of shaft material is 40×10^3 kg/m³, therefore mass of the shaft per metre length,

$$m_S = \text{Area} \times \text{length} \times \text{density} = \frac{\pi}{4}(0.02)^2 \times 1 \times 40 \times 10^3 = 12.6 \text{ kg/m}$$

We know that static deflection due to 1 kg of mass at the centre,

$$\delta = \frac{Wl^3}{48\,EI} = \frac{1 \times 9.81 (0.6)^3}{48 \times 200 \times 10^9 \times 7.855 \times 10^{-9}} = 28 \times 10^{-6} \text{ m}$$

and static deflection due to mass of the shaft,

$$\delta_S = \frac{5\,wl^4}{384\,EI} = \frac{5 \times 12.6 \times 9.81 (0.6)^4}{384 \times 200 \times 10^9 \times 7.855 \times 10^{-9}} = 0.133 \times 10^{-3} \text{ m}$$

$\therefore$ Frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta + \frac{\delta_S}{1.27}}} + \frac{0.4985}{\sqrt{28 \times 10^{-6} + \frac{0.133 \times 10^{-3}}{1.27}}}$$

$$= \frac{0.4985}{11.52 \times 10^{-3}} = 43.3 \text{ Hz}$$

Let N_c = Whirling speed of a shaft.

We know that whirling speed of a shaft in r.p.s. is equal to the frequency of transverse vibration in Hz , therefore

N_c = 43.3 r.p.s. = 43.3 × 60 = 2598 r.p.m. **Ans.**

Example 23.6. *A shaft 1.5 m long, supported in flexible bearings at the ends carries two wheels each of 50 kg mass. One wheel is situated at the centre of the shaft and the other at a distance of 375 mm from the centre towards left. The shaft is hollow of external diameter 75 mm and internal diameter 40 mm. The density of the shaft material is 7700 kg/m³ and its modulus of elasticity is 200 GN/m². Find the lowest whirling speed of the shaft, taking into account the mass of the shaft.*

Solution. l = 1.5 m ; $m_1 = m_2$ = 50 kg ; d_1 = 75 mm = 0.075 m ; d_2 = 40 mm = 0.04 m ; ρ = 7700 kg/m³ ; E = 200 GN/m² = 200 × 10^9 N/m²

The shaft is shown in Fig. 23.16.

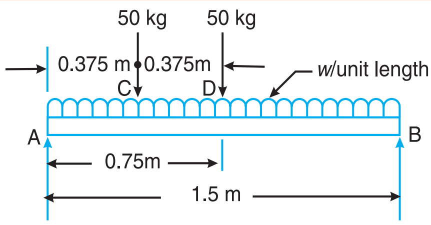


Fig. 23.16

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64}\left[(d_1)^4 - (d_2)^4\right] = \frac{\pi}{64}\left[(0.075)^4 - (0.04)^4\right] = 1.4 \times 10^{-6} \text{ m}^4$$

Since the density of shaft material is 7700 kg/m³, therefore mass of the shaft per metre length,

$$m_S = \text{Area} \times \text{length} \times \text{density}$$

$$= \frac{\pi}{4}\left[(0.075)^2 - (0.04)^2\right] 1 \times 7700 = 24.34 \text{ kg/m}$$

We know that the static deflection due to a load W

$$= \frac{Wa^2b^2}{3EIl} = \frac{m.ga^2b^2}{3EIl}$$

$\therefore$ Static deflection due to a mass of 50 kg at C,

$$\delta_1 = \frac{m_1 ga^2b^2}{3EIl} = \frac{50 \times 9.81 (0.375)^2 (1.125)^2}{3 \times 200 \times 10^9 \times 1.4 \times 10^{-6} \times 1.5} = 70 \times 10^{-6} \text{ m}$$

. . . (Here a = 0.375 m, and b = 1.125 m)

Similarly, static deflection due to a mass of 50 kg at D

$$\delta_2 = \frac{m_1 ga^2b^2}{3EIl} = \frac{50 \times 9.81 (0.75)^2 (0.75)^2}{3 \times 200 \times 10^9 \times 1.4 \times 10^{-6} \times 1.5} = 123 \times 10^{-6} \text{ m}$$

. . . (Here $a = b$ = 0.75 m)

We know that static deflection due to uniformly distributed load or mass of the shaft,

$$\delta_S = \frac{5}{384} \times \frac{wl^4}{EI} = \frac{5}{384} \times \frac{24.34 \times 9.81 (1.5)^4}{200 \times 10^9 \times 1.4 \times 10^{-6}} = 56 \times 10^{-6} \text{ m}$$

. . . (Substituting, $w = m_S \times g$)

We know that frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \dfrac{\delta_S}{1.27}}} = \frac{0.4985}{\sqrt{70 \times 10^{-6} + 123 \times 10^{-6} + \dfrac{56 \times 10^{-6}}{1.27}}} \text{ Hz}$$

$$= 32.4 \text{ Hz}$$

Since the whirling speed of shaft (N_c) in r.p.s. is equal to the frequency of transverse vibration in Hz, therefore

$$N_c = 32.4 \text{ r.p.s.} = 32.4 \times 60 = 1944 \text{ r.p.m.}$$ **Ans.**

Example 23.7. *A vertical shaft of 5 mm diameter is 200 mm long and is supported in long bearings at its ends. A disc of mass 50 kg is attached to the centre of the shaft. Neglecting any increase in stiffness due to the attachment of the disc to the shaft, find the critical speed of rotation and the maximum bending stress when the shaft is rotating at 75% of the critical speed. The centre of the disc is 0.25 mm from the geometric axis of the shaft. E = 200 GN/m*^{2}.

Solution. Given : d = 5 mm = 0.005 m ; l = 200 mm = 0.2 m ; m = 50 kg ; e = 0.25 mm = 0.25×10^{-3} m ; E = 200 GN/m^2 = 200×10^9 N/m^2

Critical speed of rotation

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.005)^4 = 30.7 \times 10^{-12} \text{ m}^4$$

Since the shaft is supported in long bearings, it is assumed to be fixed at both ends. We know that the static deflection at the centre of the shaft due to a mass of 50 kg,

$$\delta = \frac{Wl^3}{192\, EI} = \frac{50 \times 9.81 (0.2)^3}{192 \times 200 \times 10^9 \times 30.7 \times 10^{-12}} = 3.33 \times 10^{-3} \text{ m}$$

. . . ($\because$ $W = m.g$)

We know that critical speed of rotation (or natural frequency of transverse vibrations),

$$N_c = \frac{0.4985}{\sqrt{3.33 \times 10^{-3}}} = 8.64 \text{ r.p.s.}$$ **Ans.**

Maximum bending stress

Let σ = Maximum bending stress in N/m^2, and

N = Speed of the shaft = 75% of critical speed = 0.75 N_c . . . (Given)

When the shaft starts rotating, the additional dynamic load (W_1) to which the shaft is subjected, may be obtained by using the bending equation,

$$\frac{M}{I} = \frac{\sigma}{y_1} \quad \text{or} \quad M = \frac{\sigma . I}{y_1}$$

We know that for a shaft fixed at both ends and carrying a point load (W_1) at the centre, the maximum bending moment

$$M = \frac{W_1 . l}{8}$$

$$\therefore \quad \frac{W_1 . l}{8} = \frac{\sigma \cdot I}{d/2} \qquad \ldots (\because\ y_1 = d/2)$$

and
$$W_1 = \frac{\sigma . I}{d/2} \times \frac{8}{l} = \frac{\sigma \times 30.7 \times 10^{-12}}{0.005/2} \times \frac{8}{0.2} = 0.49 \times 10^{-6}\,\sigma\,\text{N}$$

$\therefore$ Additional deflection due to load W_1,

$$y = \frac{W_1}{W} \times \delta = \frac{0.49 \times 10^{-6}\,\sigma}{50 \times 9.81} \times 3.33 \times 10^{-3} = 3.327 \times 10^{-12}\,\sigma$$

We know that

$$y = \frac{\pm e}{\left(\frac{\omega_c}{\omega}\right)^2 - 1} = \frac{\pm e}{\left(\frac{N_c}{N}\right) - 1} \qquad \ldots (\text{Substituting } \omega_c = N_c \text{ and } \omega = N)$$

$$3.327 \times 10^{-12}\,\sigma = \frac{\pm 0.25 \times 10^{-3}}{\left(\frac{N_c}{0.75\,N_c}\right)^2 - 1} = \pm 0.32 \times 10^{-3}$$

$$\sigma = 0.32 \times 10^{-3} / 3.327 \times 10^{-12} = 0.0962 \times 10^{9}\ \text{N/m}^2 \quad \ldots (\text{Taking + ve sign})$$

$= 96.2 \times 10^6$ N/m^2 = 96.2 MN/m^2 **Ans.**

Example 23.8. *A vertical steel shaft 15 mm diameter is held in long bearings 1 metre apart and carries at its middle a disc of mass 15 kg. The eccentricity of the centre of gravity of the disc from the centre of the rotor is 0.30 mm.*

The modulus of elasticity for the shaft material is 200 GN/m^2 and the permissible stress is 70 MN/m^2. Determine : **1.** *The critical speed of the shaft and* **2.** *The range of speed over which it is unsafe to run the shaft. Neglect the mass of the shaft.*

[*For a shaft with fixed end carrying a concentrated load (W) at the centre assume* $\delta = \frac{Wl^3}{192\,EI}$, *and* $M = \frac{W.l}{8}$, *where* δ *and M are maximum deflection and bending moment respectively*].

Solution. Given : d = 15 mm = 0.015 m ; l = 1 m ; m = 15 kg ; e = 0.3 mm = 0.3 × 10^{-3} m ; E = 200 GN/m^2 = 200 × 10^9 N/m^2 ; σ = 70 MN/m^2 = 70 × 10^6 N/m^2

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.015)^4 = 2.5 \times 10^{-9}\,\text{m}^4$$

1. Critical speed of the shaft

Since the shaft is held in long bearings, therefore it is assumed to be fixed at both ends. We know that the static deflection at the centre of shaft,

$$\delta = \frac{Wl^3}{192\,EI} = \frac{15 \times 9.81 \times 1^3}{192 \times 200 \times 10^9 \times 2.5 \times 10^{-9}} = 1.5 \times 10^{-3}\,\text{m} \qquad \ldots (\because\ W = m.g)$$

∴ Natural frequency of transverse vibrations,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{1.5\times10^{-3}}} = 12.88\,\text{Hz}$$

We know that the critical speed of the shaft in r.p.s. is equal to the natural frequency of transverse vibrations in Hz.

∴ Critical speed of the shaft,

N_c = 12.88 r.p.s. = 12.88 × 60 = 772.8 r.p.m. **Ans.**

2. ***Range of speed***

Let N_1 and N_2 = Minimum and maximum speed respectively.

When the shaft starts rotating, the additional dynamic load ($W_1 = m_1.g$) to which the shaft is subjected may be obtained from the relation

$$\frac{M}{I} = \frac{\sigma}{y_1} \quad \text{or} \quad M = \frac{\sigma.I}{y_1}$$

Since $M = \dfrac{W_1.l}{8} = \dfrac{m_1.g.l}{8}$, and $y_1 = \dfrac{d}{2}$, therefore

$$\frac{m_1.g.l}{8} = \frac{\sigma.I}{d/2}$$

or $$m_1 = \frac{8\times2\times\sigma\times I}{d.g.l} = \frac{8\times2\times70\times10^6\times2.5\times10^{-9}}{0.015\times9.81\times1} = 19\text{ kg}$$

∴ Additional deflection due to load $W_1 = m_1 g$,

$$y = \frac{W_1}{W}\times\delta = \frac{m_1}{m}\times\delta = \frac{19}{15}\times1.5\times10^{-3} = 1.9\times10^{-3}\,\text{m}$$

We know that,

$$y = \frac{\pm e}{\left(\frac{\omega_c}{\omega}\right)^2 - 1} \quad \text{or} \quad \pm\frac{y}{e} = \frac{1}{\left(\frac{N_c}{N}\right)^2 - 1}$$

. . . (Substituting, $\omega_c = N_c$, and $\omega = N$)

$$\therefore \quad \pm\frac{1.9\times10^{-3}}{0.3\times10^{-3}} = \frac{1}{\left(\frac{N_c}{N}\right)^2 - 1} \quad \text{or} \quad \left(\frac{N_c}{N}\right)^2 - 1 = \pm\frac{0.3}{1.9} = \pm0.16$$

$$\left(\frac{N_c}{N}\right)^2 = 1 \pm 0.16 = 1.16 \quad \text{or} \quad 0.84$$

. . . (Taking first plus sign and then negative sign)

or $$N = \frac{N_c}{\sqrt{1.16}} \quad \text{or} \quad \frac{N_c}{\sqrt{0.84}}$$

$$\therefore \qquad N_1 = \frac{N_c}{\sqrt{1.16}} = \frac{772.8}{\sqrt{1.16}} = 718 \text{ r.p.m.}$$

and
$$N_2 = \frac{N_c}{\sqrt{0.84}} = \frac{772.8}{\sqrt{0.84}} = 843 \text{ r.p.m.}$$

Hence the range of speed is from 718 r.p.m. to 843 r.p.m. **Ans.**

23.13. Frequency of Free Damped Vibrations (Viscous Damping)

We have already discussed that the motion of a body is resisted by frictional forces. In vibrating systems, the effect of friction is referred to as damping. The damping provided by fluid resistance is known as ***viscous damping.***

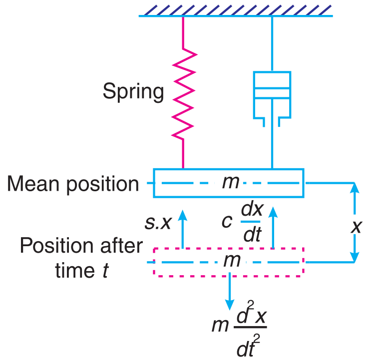


Fig. 23.17. Frequency of free damped vibrations.

We have also discussed that in damped vibrations, the amplitude of the resulting vibration gradually diminishes. This is due to the reason that a certain amount of energy is always dissipated to overcome the frictional resistance. The resistance to the motion of the body is provided partly by the medium in which the vibration takes place and partly by the internal friction, and in some cases partly by a dash pot or other external damping device.

Consider a vibrating system, as shown in Fig. 23.17, in which a mass is suspended from one end of the spiral spring and the other end of which is fixed. A damper is provided between the mass and the rigid support.

Let m = Mass suspended from the spring,

s = Stiffness of the spring,

x = Displacement of the mass from the mean position at time t,

δ = Static deflection of the spring

= $m.g/s$, and

c = Damping coefficient or the damping force per unit velocity.

Since in viscous damping, it is assumed that the frictional resistance to the motion of the body is directly proportional to the speed of the movement, therefore

Damping force or frictional force on the mass acting in ***opposite*** direction to the motion of the mass

$$= c \times \frac{dx}{dt}$$

Accelerating force on the mass, acting ***along*** the motion of the mass

$$= m \times \frac{d^2x}{dt^2}$$

Riveting Machine

Note : This picture is given as additional information and is not a direct example of the current chapter.

and spring force on the mass, acting in ***opposite*** direction to the motion of the mass,

$$= s.x$$

Therefore the equation of motion becomes

$$m \times \frac{d^2x}{dt^2} = -\left(c \times \frac{dx}{dt} + s.x\right)$$

...(Negative sign indicates that the force opposes the motion)

or $$m \times \frac{d^2x}{dt^2} + c \times \frac{dx}{dt} + s.x = 0$$

or * $$\frac{d^2x}{dt^2} + \frac{c}{m} \times \frac{dx}{dt} + \frac{s}{m} \times x = 0$$

This is a differential equation of the second order. Assuming a solution of the form $x = e^{kt}$ where k is a constant to be determined. Now the above differential equation reduces to

$$k^2.e^{kt} + \frac{c}{m} \times k.e^{kt} + \frac{s}{m} \times e^{kt} = 0 \qquad \ldots\left[\because \frac{dx}{dt} = ke^{kt}, \text{ and } \frac{d^2x}{dt^2} = k^2.e^{kt}\right]$$

or $$k^2 + \frac{c}{m} \times k + \frac{s}{m} = 0 \qquad \ldots (i)$$

and $$k = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4 \times \frac{s}{m}}}{2}$$

$$= -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}$$

$\therefore$ The two roots of the equation are

$$k_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}$$

and $$k_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}$$

The most general solution of the differential equation ***(i)*** with its right hand side equal to zero has only complementary function and it is given by

$$x = C_1 e^{k_1 t} + C_2\, e^{k_2 t} \qquad \ldots (ii)$$

where C_1 and C_2 are two arbitrary constants which are to be determined from the initial conditions of the motion of the mass.

It may be noted that the roots k_1 and k_2 may be real, complex conjugate (imaginary) or equal. We shall now discuss these three cases as below :

---

* A system described by this equation is said to be a single degree of freedom harmonic oscillator with viscous damping.

1. *When the roots are real (overdamping)*

If $\left(\frac{c}{2m}\right)^2 > \frac{s}{m}$, then the roots k_1 and k_2 are real but negative. This is a case of ***overdamping*** or ***large damping*** and the mass moves slowly to the equilibrium position. This motion is known as ***aperiodic.*** When the roots are real, the most general solution of the differential equation is

$$x = C_1 e^{k_1 t} + C_2 e^{k_2 t}$$

$$= C_1\, e^{\left[-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}\right]t} + C_2\, e^{\left[-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}\right]t}$$

Note : In actual practice, the overdamped vibrations are avoided.

2. *When the roots are complex conjugate (underdamping)*

If $\frac{s}{m} > \left(\frac{c}{2m}\right)^2$, then the radical (*i.e.* the term under the square root) becomes negative. The two roots k_1 and k_2 are then known as complex conjugate. This is a most practical case of damping and it is known as ***underdamping*** or ***small damping.*** The two roots are

$$k_1 = -\frac{c}{2m} + i\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}$$

and
$$k_2 = -\frac{c}{2m} - i\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}$$

where i is a Greek letter known as iota and its value is $\sqrt{-1}$. For the sake of mathematical calculations, let

$$\frac{c}{2m} = a;\ \frac{s}{m} = (\omega_n)^2;\ \text{and}\ \sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2} = \omega_d = \sqrt{(\omega_n)^2 - a^2}$$

Therefore the two roots may be written as

$$k_1 = -a + i\,\omega_d\ ; \quad \text{and} \quad k_2 = -a - i\,\omega_d$$

We know that the general solution of a differential equation is

$$x = C_1\, e^{k_1 t} + C_2\, e^{k_2 t} = C_1\, e^{(-a + i\omega_d)t} + C_2\, e^{(-a - i\omega_d)t}$$

$$= e^{-at}(C_1\, e^{i\omega_d . t} + C_2\, e^{-i\omega_d t}) \quad \ldots(\text{Using } e^{m+n} = e^m \times e^n) \ \ldots(iii)$$

Now according to Euler's theorem

$$e^{+i\theta} = \cos\theta + i\sin\theta\ ;\ \text{and}\ e^{-i\theta} = \cos\theta - i\sin\theta$$

Therefore the equation ***(iii)*** may be written as

$$x = e^{-at}\left[C_1(\cos\omega_d .t + i\sin\omega_d .t) + C_2(\cos\omega_d .t - i\sin\omega_d .t)\right]$$

$$= e^{-at}\left[(C_1 + C_2)\cos\omega_d .t + i\,(C_1 - C_2)\sin\omega_d .t\right]$$

Let
$$C_1 + C_2 = A,\ \text{and}\ i\,(C_1 - C_2) = B$$

$\therefore \qquad x = e^{-at}(A\cos\omega_d.t + B\sin\omega_d.t)$ . . . (*iv*)

Again, let $A = C\cos\theta$, and $B = C\sin\theta$, therefore

$$C = \sqrt{A^2 + B^2}\ , \text{ and } \quad \tan\theta = \frac{B}{A}$$

Now the equation (*iv*) becomes

$$x = e^{-at}(C\cos\theta\cos\omega_d.t + C\sin\theta\sin\omega_d.t)$$

$$= Ce^{-at}\cos(\omega_d.t - \theta) \qquad \text{. . . (v)}$$

If t is measured from the instant at which the mass m is released after an initial displacement A, then

$A = C\cos\theta$ . . . [Substituting $x = A$ and $t = 0$ in equation (*v*)]

and when $\theta = 0$, then $A = C$

$\therefore$ The equation (*v*) may be written as

$$x = Ae^{-at}\cos\omega_d.t \qquad \text{. . . (vi)}$$

where $\omega_d = \sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2} = \sqrt{(\omega_n)^2 - a^2}$; and $a = \frac{c}{2m}$

We see from equation (*vi*), that the motion of the mass is simple harmonic whose circular damped frequency is ω_d and the amplitude is Ae^{-at} which diminishes exponentially with time as shown in Fig. 23.18. Though the mass eventually returns to its equilibrium position because of its inertia, yet it overshoots and the oscillations may take some considerable time to die away.

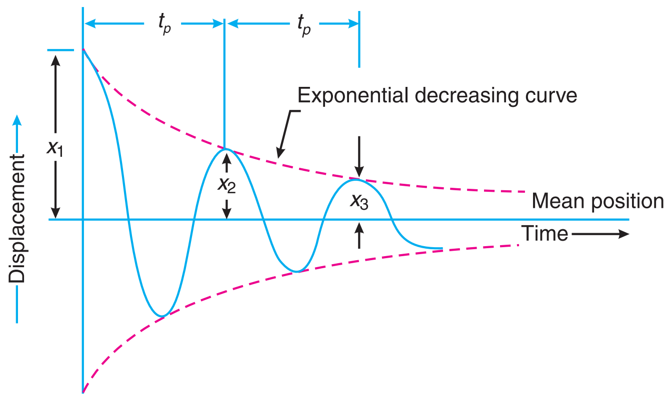


Fig. 23.18. Underdamping or small damping.

We know that the periodic time of vibration,

$$t_p = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}} = \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

and frequency of damped vibration,

$$f_d = \frac{1}{t_p} = \frac{\omega_d}{2\pi} = \frac{1}{2\pi}\sqrt{(\omega_n)^2 - a^2} = \frac{1}{2\pi}\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2} \qquad \text{. . . (vii)}$$

Note : When no damper is provided in the system, then $c = 0$. Therefore the frequency of the undamped vibration,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{s}{m}}$$

. . . [Substituting $c = 0$, in equation (*vii*)]

It is the same as discussed under free vibrations.

3. ***When the roots are equal (critical damping)***

If $\left(\frac{c}{2m}\right)^2 = \frac{s}{m}$, then the radical becomes zero and the two roots k_1 and k_2 are equal. This is a case of **critical damping.** In other words, the critical damping is said to occur when frequency of damped vibration (f_d) is zero (*i.e.* motion is aperiodic). This type of damping is also avoided because the mass moves back rapidly to its equilibrium position, in the shortest possible time.

In a disc brake, hydraulic pressure forces friction pads to squeeze a metal disc that rotates on the same axle as the wheel. Here a disc brake is being tested.

Note : This picture is given as additional information and is not a direct example of the current chapter.

For critical damping, equation (*ii*) may be written as

$$x = (C_1 + C_2)\, e^{-\frac{c}{2m}t} = (C_1 + C_2)\, e^{-\omega_n t} \qquad \ldots \left[\because \frac{c}{2m} = \sqrt{\frac{s}{m}} = \omega_n\right]$$

Thus the motion is again aperiodic. The critical damping coefficient (c_c) may be obtained by substituting c_c for c in the condition for critical damping, *i.e.*

$$\left(\frac{c_c}{2m}\right)^2 = \frac{s}{m} \qquad \text{or} \qquad c_c = 2m\sqrt{\frac{s}{m}} = 2m \times \omega_n$$

The critical damping coefficient is the amount of damping required for a system to be critically damped.

23.14. Damping Factor or Damping Ratio

The ratio of the actual damping coefficient (c) to the critical damping coefficient (c_c) is known as ***damping factor* or *damping ratio*.** Mathematically,

$$\text{Damping factor} = \frac{c}{c_c} = \frac{c}{2m.\omega_n} \qquad \ldots (\because c_c = 2\pi.\omega_n)$$

The damping factor is the measure of the relative amount of damping in the existing system with that necessary for the critical damped system.

23.15. Logarithmic Decrement

It is defined as the natural logarithm of the amplitude reduction factor. The amplitude reduction factor is the ratio of any two successive amplitudes on the same side of the mean position.

If x_1 and x_2 are successive values of the amplitude on the same side of the mean position, as shown in Fig. 23.18, then amplitude reduction factor,

$$\frac{x_1}{x_2} = \frac{Ae^{-at}}{Ae^{-a(t+t_p)}} = e^{at_p} = \text{constant}$$

where t_p is the period of forced oscillation or the time difference between two consecutive amplitudes. As per definition, logarithmic decrement,

$$\delta = \log\left(\frac{x_1}{x_2}\right) = \log e^{at_p}$$

or

$$\delta = \log_e\left(\frac{x_1}{x_2}\right) = a.t_p = a \times \frac{2\pi}{\omega_d} = \frac{a \times 2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

$$\ldots \left[\because \omega_d = \sqrt{(\omega_n)^2 - a^2}\right]$$

$$= \frac{\frac{c}{2m} \times 2\pi}{\sqrt{(\omega_n)^2 - \left(\frac{c}{2m}\right)^2}} \qquad \ldots \left(\because a = \frac{c}{2m}\right)$$

$$= \frac{\frac{c}{2m} \times 2\pi}{\omega_n \sqrt{1 - \left(\frac{c}{2m.\omega_n}\right)^2}} = \frac{c \times 2\pi}{c_c \sqrt{1 - \left(\frac{c}{c_c}\right)^2}} \qquad \ldots (\because c_c = 2m.\omega_n)$$

$$= \frac{2\pi \times c}{\sqrt{(c_c)^2 - c^2}}$$

In general, amplitude reduction factor,

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \ldots = \frac{x_n}{x_{n+1}} = e^{at_p} = \text{constant}$$

∴ Logarithmic decrement,

$$\delta = \log_e\left(\frac{x_n}{x_{n+1}}\right) = a.t_p = \frac{2\pi \times c}{\sqrt{(c_c)^2 - c^2}}$$

Example 23.9. *A vibrating system consists of a mass of 200 kg, a spring of stiffness 80 N/mm and a damper with damping coefficient of 800 N/m/s. Determine the frequency of vibration of the system.*

Solution. Given : m = 200 kg ; s = 80 N/mm = 80×10^3 N/m ; c = 800 N/m/s

We know that circular frequency of undamped vibrations,

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{80 \times 10^3}{200}} = 20 \text{ rad/s}$$

and circular frequency of damped vibrations,

$$\omega_d = \sqrt{(\omega_n)^2 - a^2} = \sqrt{(\omega_n)^2 - (c/2m)^2} \qquad \ldots (\because\ a = c/2m)$$

$$= \sqrt{(20)^2 - (800/2 \times 200)^2} = 19.9 \text{ rad/s}$$

∴ Frequency of vibration of the system,

$$f_d = \omega_d / 2\pi = 19.9/2\pi = 3.17 \text{ Hz} \quad \textbf{Ans.}$$

Example 23.10. *The following data are given for a vibratory system with viscous damping:*

Mass = 2.5 kg ; spring constant = 3 N/mm and the amplitude decreases to 0.25 of the initial value after five consecutive cycles.

Determine the damping coefficient of the damper in the system.

Solution. Given : m = 2.5 kg ; s = 3 N/mm = 3000 N/m ; $x_6 = 0.25\ x_1$

We know that natural circular frequency of vibration,

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{3000}{2.5}} = 34.64 \text{ rad/s}$$

Let c = Damping coefficient of the damper in N/m/s,

x_1 = Initial amplitude, and

x_6 = Final amplitude after five consecutive cycles = 0.25 x_1 ...(Given)

We know that

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5} = \frac{x_5}{x_6}$$

or

$$\frac{x_1}{x_6} = \frac{x_1}{x_2} \times \frac{x_2}{x_3} \times \frac{x_3}{x_4} \times \frac{x_4}{x_5} \times \frac{x_5}{x_6} = \left(\frac{x_1}{x_2}\right)^5$$

∴

$$\frac{x_1}{x_2} = \left(\frac{x_1}{x_6}\right)^{1/5} = \left(\frac{x_1}{0.25\,x_1}\right)^{1/5} = (4)^{1/5} = 1.32$$

We know that

$$\log_e\left(\frac{x_1}{x_2}\right) = a \times \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

$$\log_e(1.32) = a \times \frac{2\pi}{\sqrt{(34.64)^2 - a^2}} \quad \text{or} \quad 0.2776 = \frac{a \times 2\pi}{\sqrt{1200 - a^2}}$$

Squaring both sides,

$$0.077 = \frac{39.5\,a^2}{1200 - a^2} \quad \text{or} \quad 92.4 - 0.077\,a^2 = 39.5\,a^2$$

∴ $a^2 = 2.335$ or a = 1.53

We know that $a = c/2m$ or $c = a \times 2m$ = 1.53 × 2 × 2.5 = 7.65 N/m/s **Ans.**

Example 23.11. *An instrument vibrates with a frequency of 1 Hz when there is no damping. When the damping is provided, the frequency of damped vibrations was observed to be 0.9 Hz. Find **1.** the damping factor, and **2.** logarithmic decrement.*

Guitar

Solution. Given : f_n = 1 Hz ; f_d = 0.9 Hz

1. ***Damping factor***

Let m = Mass of the instrument in kg,

c = Damping coefficient or damping force per unit velocity in N/m/s, and

c_c = Critical damping coefficient in N/m/s.

We know that natural circular frequency of undamped vibrations,

$$\omega_n = 2\pi \times f_n = 2\pi \times 1 = 6.284 \text{ rad/s}$$

and circular frequency of damped vibrations,

$$\omega_d = 2\pi \times f_d = 2\pi \times 0.9 = 5.66 \text{ rad/s}$$

We also know that circular frequency of damped vibrations (ω_d),

$$5.66 = \sqrt{(\omega_n)^2 - a^2} = \sqrt{(6.284)^2 - a^2}$$

Squaring both sides,

$$(5.66)^2 = (6.284)^2 - a^2 \text{ or } 32 = 39.5 - a^2$$

$\therefore$ $a^2 = 7.5$ or $a = 2.74$

We know that, $a = c/2m$ or $c = a \times 2m = 2.74 \times 2m = 5.48\, m$ N/m/s

and $c_c = 2m.\omega_n = 2m \times 6.284 = 12.568\, m$ N/m/s

$\therefore$ Damping factor,

$$c / c_c = 5.48m / 12.568m = 0.436 \textbf{ Ans.}$$

2. ***Logarithmic decrement***

We know that logarithmic decrement,

$$\delta = \frac{2\pi c}{\sqrt{(c_c)^2 - c^2}} = \frac{2\pi \times 5.48\, m}{\sqrt{(12.568\, m)^2 - (5.48\, m)^2}} = \frac{34.4}{11.3} = 3.04 \textbf{ Ans.}$$

Example 23.12. *The measurements on a mechanical vibrating system show that it has a mass of 8 kg and that the springs can be combined to give an equivalent spring of stiffness 5.4 N/mm. If the vibrating system have a dashpot attached which exerts a force of 40 N when the mass has a velocity of 1 m/s, find : **1.** critical damping coefficient, **2.** damping factor, **3.** logarithmic decrement, and **4.** ratio of two consecutive amplitudes.*

Solution. Given : m = 8 kg ; s = 5.4 N/mm = 5400 N/m

Since the force exerted by dashpot is 40 N, and the mass has a velocity of 1 m/s , therefore

Damping coefficient (actual),

$$c = 40 \text{ N/m/s}$$

1. *Critical damping coefficient*

We know that critical damping coefficient,

$$c_c = 2m.\omega_n = 2m \times \sqrt{\frac{s}{m}} = 2 \times 8\sqrt{\frac{5400}{8}} = 416 \text{ N/m/s } \textbf{Ans.}$$

2. *Damping factor*

We know that damping factor

$$= \frac{c}{c_c} = \frac{40}{416} = 0.096 \textbf{ Ans.}$$

3. *Logarithmic decrement*

We know that logarithmic decrement,

$$\delta = \frac{2\pi c}{\sqrt{(c_c)^2 - c^2}} = \frac{2\pi \times 40}{\sqrt{(416)^2 - (40)^2}} = 0.6 \textbf{ Ans.}$$

4. *Ratio of two consecutive amplitudes*

Let x_n and x_{n+1} = Magnitude of two consecutive amplitudes,

We know that logarithmic decrement,

$$\delta = \log_e \left[\frac{x_n}{x_{n+1}}\right] \text{ or } \frac{x_n}{x_{n+1}} = e^{\delta} = (2.7)^{0.6} = 1.82 \textbf{ Ans.}$$

Example 23.13. *A mass suspended from a helical spring vibrates in a viscous fluid medium whose resistance varies directly with the speed. It is observed that the frequency of damped vibration is 90 per minute and that the amplitude decreases to 20 % of its initial value in one complete vibration. Find the frequency of the free undamped vibration of the system.*

Helical spring suspension of a two-wheeler.

Note : This picture is given as additional information and is not a direct example of the current chapter.

Solution. Given : f_d = 90/min = 90/60 = 1.5 Hz

We know that time period,

$t_p = 1/f_d = 1/1.5 = 0.67$ s

Let x_1 = Initial amplitude, and

x_2 = Final amplitude after one complete vibration

$= 20\% \; x_1 = 0.2 \; x_1$ . . . (Given)

We know that

$$\log_e\left(\frac{x_1}{x_2}\right) = a.t_p \quad \text{or} \quad \log_e\left(\frac{x_1}{0.2\,x_1}\right) = a \times 0.67$$

∴ $\log_e 5 = 0.67\, a$ or $1.61 = 0.67\, a$ or $a = 2.4$ . . . ($\because \log_e 5 = 1.61$)

We also know that frequency of free damped vibration,

$$f_d = \frac{1}{2\pi}\sqrt{(\omega_n)^2 - a^2}$$

or $$(\omega_n)^2 = (2\pi \times f_d)^2 + a^2 \quad \text{. . . (By squaring and arranging)}$$

$$= (2\pi \times 1.5)^2 + (2.4)^2 = 94.6$$

∴ $$\omega_n = 9.726 \text{ rad/s}$$

We know that frequency of undamped vibration,

$$f_n = \frac{\omega_n}{2\pi} = \frac{9.726}{2\pi} = 1.55 \text{ Hz } \textbf{Ans.}$$

Example 23.14. *A coil of spring stiffness 4 N/mm supports vertically a mass of 20 kg at the free end. The motion is resisted by the oil dashpot. It is found that the amplitude at the beginning of the fourth cycle is 0.8 times the amplitude of the previous vibration. Determine the damping force per unit velocity. Also find the ratio of the frequency of damped and undamped vibrations.*

Solution. Given : s = 4 N/mm = 4000 N/m ; m = 20 kg

Damping force per unit velocity

Let c = Damping force in newtons per unit velocity *i.e.* in N/m/s

x_n = Amplitude at the beginning of the third cycle,

x_{n+1} = Amplitude at the beginning of the fourth cycle = 0.8 x_n . . . (Given)

We know that natural circular frequency of motion,

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{4000}{20}} = 14.14 \text{ rad/s}$$

and $$\log_e\left(\frac{x_n}{x_n + 1}\right) = a \times \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

or $$\log_e\left(\frac{x_n}{0.8\,x_n}\right) = a \times \frac{2\pi}{\sqrt{(14.14)^2 - a^2}}$$

$$\log_e 1.25 = a \times \frac{2\pi}{\sqrt{200 - a^2}} \quad \text{or} \quad 0.223 = a \times \frac{2\pi}{\sqrt{200 - a^2}}$$

Squaring both sides

$$0.05 = \frac{a^2 \times 4\pi^2}{200 - a^2} = \frac{39.5\,a^2}{200 - a^2}$$

$0.05 \times 200 - 0.05\,a^2 = 39.5a^2$ or $39.55\,a^2 = 10$

∴ $a^2 = 10 / 39.55 = 0.25$ or $a = 0.5$

We know that $a = c / 2m$

∴ $c = a \times 2m = 0.5 \times 2 \times 20 = 20$ N/m/s **Ans.**

Ratio of the frequencies

Let f_{n1} = Frequency of damped vibrations = $\dfrac{\omega_d}{2\pi}$

f_{n2} = Frequency of undamped vibrations = $\frac{\omega_n}{2\pi}$

$\therefore$

$$\frac{f_{n1}}{f_{n2}} = \frac{\omega_d}{2\pi} \times \frac{2\pi}{\omega_n} = \frac{\omega_d}{\omega_n} = \sqrt{\frac{(\omega_n)^2 - a^2}{\omega_n}} = \sqrt{\frac{(14.14)^2 - (0.5)^2}{14.14}}$$

$$\ldots \left(\because \ \omega_d = \sqrt{(\omega_n)^2 - a^2}\right)$$

$= 0.999$ **Ans.**

Example 23.15. *A machine of mass 75 kg is mounted on springs and is fitted with a dashpot to damp out vibrations. There are three springs each of stiffness 10 N/mm and it is found that the amplitude of vibration diminishes from 38.4 mm to 6.4 mm in two complete oscillations. Assuming that the damping force varies as the velocity, determine :* **1.** *the resistance of the dash-pot at unit velocity ;* **2.** *the ratio of the frequency of the damped vibration to the frequency of the undamped vibration ; and* **3.** *the periodic time of the damped vibration.*

Solution. Given : $m = 75$ kg ; $s = 10$ N/mm $= 10 \times 10^3$ N/m ; $x_1 = 38.4$ mm $= 0.0384$ m ; $x_3 = 6.4$ mm $= 0.0064$ m

Since the stiffness of each spring is 10×10^3 N/m and there are 3 springs, therefore total stiffness,

$$s = 3 \times 10 \times 10^3 = 30 \times 10^3 \text{ N/m}$$

We know that natural circular frequency of motion,

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{30 \times 10^3}{75}} = 20 \text{ rad/s}$$

1. Resistance of the dashpot at unit velocity

Let c = Resistance of the dashpot in newtons at unit velocity *i.e.* in N/m/s,

x_2 = Amplitude after one complete oscillation in metres, and

x_3 = Amplitude after two complete oscillations in metres.

We know that $\frac{x_1}{x_2} = \frac{x_2}{x_3}$

$$\therefore \quad \left(\frac{x_1}{x_2}\right)^2 = \frac{x_1}{x_3} \qquad \ldots \left[\because \ \frac{x_1}{x_3} = \frac{x_1}{x_2} \times \frac{x_2}{x_3} = \frac{x_1}{x_2} \times \frac{x_1}{x_2} = \left(\frac{x_1}{x_2}\right)^2\right]$$

or

$$\frac{x_1}{x_2} = \left(\frac{x_1}{x_3}\right)^{1/2} = \left(\frac{0.0384}{0.0064}\right)^{1/2} = 2.45$$

We also know that

$$\log_e\left(\frac{x_1}{x_2}\right) = a \times \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

$$\log_e 2.45 = a \times \frac{2\pi}{\sqrt{(20)^2 - a^2}}$$

$$0.8951 = \frac{a \times 2\pi}{\sqrt{400 - a^2}} \quad \text{or} \quad 0.8 = \frac{a^2 \times 39.5}{400 - a^2} \quad \text{. . . (Squaring both sides)}$$

∴ $a^2 = 7.94$ or $a = 2.8$

We know that $a = c / 2m$

∴ $c = a \times 2m = 2.8 \times 2 \times 75 = 420$ N/m/s **Ans.**

2. Ratio of the frequency of the damped vibration to the frequency of undamped vibration

Let f_{n1} = Frequency of damped vibration = $\dfrac{\omega_d}{2\pi}$

f_{n2} = Frequency of undamped vibration = $\dfrac{\omega_n}{2\pi}$

$$\therefore \quad \frac{f_{n1}}{f_{n2}} = \frac{\omega_d}{2\pi} \times \frac{2\pi}{\omega_n} = \frac{\omega_d}{\omega_n} = \frac{\sqrt{(\omega_n)^2 - a^2}}{\omega_n} = \frac{\sqrt{(20)^2 - (2.8)^2}}{20} = 0.99 \text{ **Ans.**}$$

3. Periodic time of damped vibration

We know that periodic time of damped vibration

$$= \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}} = \frac{2\pi}{\sqrt{(20)^2 - (2.8)^2}} = 0.32 \text{ s **Ans.**}$$

Example 23.16. *The mass of a single degree damped vibrating system is 7.5 kg and makes 24 free oscillations in 14 seconds when disturbed from its equilibrium position. The amplitude of vibration reduces to 0.25 of its initial value after five oscillations. Determine :* **1.** *stiffness of the spring,* **2.** *logarithmic decrement, and* **3.** *damping factor, i.e. the ratio of the system damping to critical damping.*

Solution. Given : $m = 7.5$ kg

Since 24 oscillations are made in 14 seconds, therefore frequency of free vibrations,

$f_n = 24/14 = 1.7$

and $\omega_n = 2\pi \times f_n = 2\pi \times 1.7 = 10.7$ rad/s

1. Stiffness of the spring

Let s = Stiffness of the spring in N/m.

We know that $(\omega_n)^2 = s / m$ or $s = (\omega_n)^2 m = (10.7)^2 7.5 = 860$ N/m **Ans.**

2. Logarithmic decrement

Let x_1 = Initial amplitude,

x_6 = Final amplitude after five oscillations = $0.25\, x_1$ ... (Given)

$$\therefore \quad \frac{x_1}{x_6} = \frac{x_1}{x_2} \times \frac{x_2}{x_3} \times \frac{x_3}{x_4} \times \frac{x_4}{x_5} \times \frac{x_5}{x_6} = \left(\frac{x_1}{x_2}\right)^5 \quad \ldots \left[\because \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5} = \frac{x_5}{x_6}\right]$$

or $$\frac{x_1}{x_2} = \left(\frac{x_1}{x_6}\right)^{1/5} = \left(\frac{x_1}{0.25\,x_1}\right)^{1/5} = (4)^{1/5} = 1.32$$

We know that logarithmic decrement,

$$\delta = \log_e\left(\frac{x_1}{x_2}\right) = \log_e 1.32 = 0.28 \text{ Ans.}$$

3. *Damping factor*

Let c = Damping coefficient for the actual system, and

c_c = Damping coefficient for the critical damped system.

We know that logarithmic decrement (δ),

$$0.28 = \frac{a \times 2\pi}{\sqrt{(\omega_n)^2 - a^2}} = \frac{a \times 2\pi}{\sqrt{(10.7)^2 - a^2}}$$

$$0.0784 = \frac{a^2 \times 39.5}{114.5 - a^2} \quad \text{. . . (Squaring both sides)}$$

$8.977 - 0.0784\,a^2 = 39.5\,a^2$ or $a^2 = 0.227$ or $a = 0.476$

We know that $a = c / 2m$ or $c = a \times 2m = 0.476 \times 2 \times 7.5 = 7.2$ N/m/s **Ans.**

and $c_c = 2m.\omega_n = 2 \times 7.5 \times 10.7 = 160.5$ N/m/s **Ans.**

$\therefore$ Damping factor $= c/c_c = 7.2 / 160.5 = 0.045$ **Ans.**

23.16. Frequency of Under Damped Forced Vibrations

Consider a system consisting of spring, mass and damper as shown in Fig. 23.19. Let the system is acted upon by an external periodic (*i.e.* simple harmonic) disturbing force,

$$F_x = F \cos \omega.t$$

where F = Static force, and

ω = Angular velocity of the periodic disturbing force.

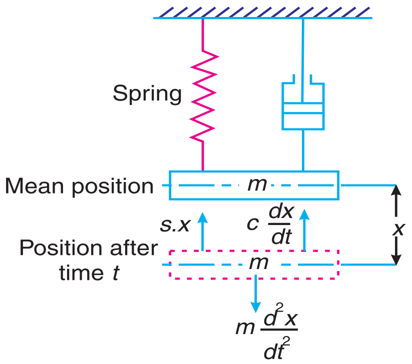


Fig. 23.19. Frequency of under damped forced vibrations.

When the system is constrained to move in vertical guides, it has only one degree of freedom. Let at sometime t, the mass is displaced downwards through a distance x from its mean position.

Using the symbols as discussed in the previous article, the equation of motion may be written as

$$m \times \frac{d^2x}{dt^2} = -c \times \frac{dx}{dt} - s.x + F \cos \omega.t$$

or $$m \times \frac{d^2x}{dt^2} + c \times \frac{dx}{dt} + s.x = F \cos \omega.t \quad \text{. . . } (i)$$

This equation of motion may be solved either by differential equation method or by graphical method as discussed below :

1. *Differential equation method*

The equation **(*i*)** is a differential equation of the second degree whose right hand side is some function in *t*. The solution of such type of differential equation consists of two parts ; one part is the complementary function and the second is particular integral. Therefore the solution may be written as

$$x = x_1 + x_2$$

where x_1 = Complementary function, and

x_2 = Particular integral.

The complementary function is same as discussed in the previous article, *i.e.*

$$x_1 = Ce^{-at}\cos(\omega_d t - \theta) \qquad \ldots (ii)$$

where C and θ are constants. Let us now find the value of particular integral as discussed below :

Let the particular integral of equation **(*i*)** is given by

$$x_2 = B_1 \sin \omega.t + B_2 \cos \omega.t \qquad \ldots \text{(where } B_1 \text{ and } B_2 \text{ are constants)}$$

$$\therefore \quad \frac{dx}{dt} = B_1.\omega\cos\omega.t - B_2.\omega\sin\omega.t$$

and

$$\frac{d^2x}{dt^2} = -B_1.\omega^2 \sin\omega.t - B_2.\omega^2\cos\omega.t$$

Substituting these values in the given differential equation **(*i*),** we get

$$m(-B_1.\omega^2 \sin\omega.t - B_2.\omega^2\cos\omega.t) + c(B_1.\omega\cos\omega.t - B_2.\omega\sin\omega.t) + s(B_1\sin\omega.t + B_2\cos\omega.t)$$
$$= F\cos\omega.t$$

or

$$(-m.B_1.\omega^2 - c.\omega.B_2 + s.B_1)\sin\omega.t + (-m.\omega^2.B_2 + c.\omega.B_1 + s.B_2)\cos\omega.t$$
$$= F\cos\omega.t$$

or

$$\left[(s - m.\omega^2)B_1 - c.\omega.B_2\right]\sin\omega.t + \left[c.\omega.B_1 + (s - m.\omega^2)B_2\right]\cos\omega.t$$
$$= F\cos\omega.t + 0\sin\omega.t$$

Comparing the coefficients of sin ω*t* and cos ω*t* on the left hand side and right hand side separately, we get

$$(s - m.\omega^2)B_1 - c.\omega.B_2 = 0 \qquad \ldots (iii)$$

and

$$c.\omega.B_1 + (s - m.\omega^2)B_2 = F \qquad \ldots (iv)$$

Now from equation **(*iii*)**

$$(s - m.\omega^2)B_1 = c.\omega.B_2$$

$$\therefore \quad B_2 = \frac{s - m.\omega^2}{c.\omega} \times B_1 \qquad \ldots (v)$$

Substituting the value of B_2 in equation **(*iv*)**

$$c.\omega.B_1 + \frac{(s - m.\omega^2)(s - m.\omega^2)}{c.\omega} \times B_1 = F$$

$$c^2.\omega^2.B_1 + (s - m.\omega^2)^2 B_1 = c.\omega.F$$

$$B_1\left[c^2.\omega^2 + (s - m.\omega^2)^2\right] = c.\omega.F$$

$\therefore$ $$B_1 = \frac{c.\omega.F}{c^2.\omega^2 + (s - m.\omega^2)^2}$$

and $$B_2 = \frac{s - m.\omega^2}{c.\omega} \times \frac{c.\omega.F}{c^2.\omega^2 + (s - m.\omega^2)^2}$$ . . . [From equation **(*v*)**]

$$= \frac{F\,(s - m.\omega^2)}{c^2.\omega^2 + (s - m.\omega^2)^2}$$

$\therefore$ The particular integral of the differential equation **(*i*)** is

$$x_2 = B_1 \sin \omega.t + B_2 \cos \omega.t$$

$$= \frac{c.\omega.F}{c^2.\omega^2 + (s - m.\omega^2)^2} \times \sin \omega.t + \frac{F\,(s - m.\omega^2)}{c^2.\omega^2 + (s - m.\omega^2)^2} \times \cos \omega.t$$

$$= \frac{F}{c^2.\omega^2 + (s - m.\omega^2)^2}\left[c.\omega \sin \omega.t + (s - m.\omega^2)\cos \omega.t\right]$$ . . . **(*vi*)**

Let $c.\omega = X \sin \phi$; and $s - m.\omega^2 = X \cos \phi$

$\therefore$ $$X = \sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}$$ . . . (By squaring and adding)

This machine performs pressing operation, welding operation and material handling.

Note : This picture is given as additional information and is not a direct example of the current chapter.

and $$\tan\phi = \frac{c.\omega}{s - m.\omega^2} \quad \text{or} \quad \phi = \tan^{-1}\left(\frac{c.\omega}{s - m.\omega^2}\right)$$

Now the equation ***(vi)*** may be written as

$$x_2 = \frac{F}{c^2.\omega^2 + (s - m.\omega^2)^2}[X \sin\phi.\sin\omega t + X \cos\phi\cos\omega t]$$

$$= \frac{F.X}{c^2.\omega^2 + (s - m.\omega^2)^2} \times \cos(\omega.t - \phi)$$

$$= \frac{F\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}}{c^2.\omega^2 + (s - m.\omega^2)^2} \times \cos(\omega.t - \phi)$$

$$= \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \times \cos(\omega.t - \phi)$$

∴ The complete solution of the differential equation ***(i)*** becomes

$$x = x_1 + x_2$$

$$= C.e^{-at}\cos(\omega_d.t - \theta) + \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \times \cos(\omega.t - \phi)$$

In actual practice, the value of the complementary function x_1 at any time t is much smaller as compared to particular integral x_2. Therefore, the displacement x, at any time t, is given by the particular integral x_2 only.

$$\therefore \quad x = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \times \cos(\omega.t - \phi) \quad \text{... (vii)}$$

This equation shows that motion is simple harmonic whose circular frequency is ω and the amplitude is $\dfrac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}}$.

A little consideration will show that the frequency of forced vibration is equal to the angular velocity of the periodic force and the amplitude of the forced vibration is equal to the maximum displacement of vibration.

∴ Maximum displacement or the amplitude of forced vibration,

$$x_{max} = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \quad \text{. . . (viii)}$$

Notes : 1. The equations ***(vii)*** and ***(viii)*** hold good when steady vibrations of constant amplitude takes place.

2. The equation ***(viii)*** may be written as

$$x_{max} = \frac{F/s}{\sqrt{\dfrac{c^2.\omega^2}{s^2} + \dfrac{(s - m.\omega^2)^2}{s^2}}}$$

. . . (Dividing the numerator and denominator by s)

$$= \frac{x_o}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left(1 - \frac{m.\omega^2}{s}\right)^2}} \qquad \text{. . . (Substituting } F/s = x_o)$$

where x_o is the deflection of the system under the static force *F*. We know that the natural frequency of free vibrations is given by

$$(\omega_n)^2 = s/m$$

$$\therefore \quad x_{max} = \frac{x_o}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}} \qquad \text{. . . (ix)}$$

3. When damping is negligible, then $c = 0$.

$$\therefore \quad x_{max} = \frac{x_o}{1 - \frac{\omega^2}{(\omega_n)^2}} = \frac{x_o(\omega_n)^2}{(\omega_n)^2 - \omega^2} = \frac{x_o \times s/m}{(\omega_n)^2 - \omega^2} \qquad \text{. . . } \left[\because (\omega_n)^2 = s/m\right]$$

$$\therefore \quad = \frac{F}{m\left[(\omega_n)^2 - \omega^2\right]} \qquad \text{. . . } (\because F = x_o.s) \text{ . . . (x)}$$

4. At resonance $\omega = \omega_n$. Therefore the angular speed at which the resonance occurs is

$$\omega = \omega_n = \sqrt{\frac{s}{m}} \text{ rad/s}$$

and
$$x_{max} = x_o \times \frac{s}{c.\omega_n} \qquad \text{. . . [From equation (ix)]}$$

2. *Graphical method*

The solution of the equation of motion for a forced and damped vibration may be easily obtained by graphical method as discussed below :

Let us assume that the displacement of the mass (*m*) in the system, as shown in Fig. 23.19, under the action of the applied simple harmonic force *F* cos ωt is itself simple harmonic, so that it can be represented by the equation,

$$x = A\cos(\omega t - \phi)$$

where *A* is the amplitude of vibration.

Now differentiating the above equation,

$$\frac{dx}{dt} = -\omega.A\sin(\omega.t - \phi) = \omega.A\cos\left[90° + (\omega.t - \phi)\right]$$

and
$$\frac{d^2x}{dt^2} = -\omega^2.A\cos(\omega.t - \phi) = \omega^2.A\cos\left[180° + (\omega.t - \phi)\right]$$

∴ Elastic force *i.e.* the force required to extend the spring

$$= s.x = s.A\cos(\omega.t - \phi)$$

Disturbing force *i.e.* the force required to overcome the resistance of dashpot

$$= c \times \frac{dx}{dt} = c.\omega.A\cos[90° + (\omega.t - \phi)]$$

and inertia force *i.e.* the force required to accelerate the mass *m*

$$= m \times \frac{d^2x}{dt^2} = m.\omega^2.A\cos[180° + (\omega.t - \phi)]$$

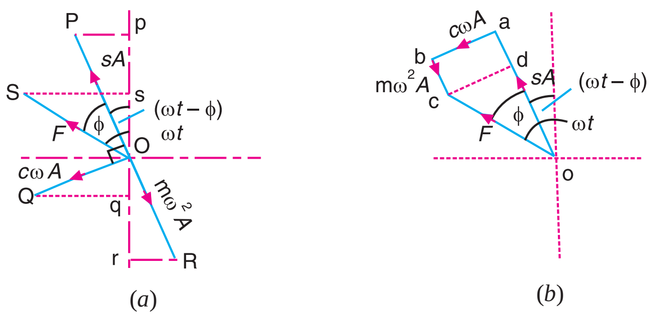


Fig. 23.20. Graphical method.

The algebraic sum of these three forces at any given instant must be equal to the applied force $F\cos\omega t$. These forces are represented graphically in Fig. 23.20 (*a*). The vector *OP* represents, to some suitable scale, the elastic force (of maximum value *s.A*), at an inclination $(\omega.t - \phi)$ to the vertical. The vector *OQ* (of maximum value $c\omega.A$) and vector *OR* (of maximum value $m.\omega^2 A$) represents, to the same scale, the disturbing force and inertia force respectively. The vectors *OP*, *OQ* and *OR* are at successive intervals of 90°.

The projected lengths *Op*, *Oq* and *Or* represent the instantaneous values of these forces at time *t* and *Os* (the algebraic sum of *Op*, *Oq* and *Or*) must represent the value *F* cos $\omega.t$ of the applied force at the same instant. Thus the force vector *OS* must be the vector sum of *OP*, *OQ* and *OR* or force *F* must be the vector sum of *s.A*, $c.\omega.A$ and $m.\omega^2.A$, as shown in Fig. 23.20 (*b*). From the geometry of the figure,

$$F = oc = \sqrt{(od)^2 + (cd)^2} = \sqrt{(oa - ad)^2 + (cd)^2}$$

$$= \sqrt{(s.A - m.\omega^2.A)^2 + (c.\omega.A)^2} = A\sqrt{(s - m.\omega^2)^2 + c^2.\omega^2}$$

∴ $$A(\text{or } x_{max}) = \frac{F}{\sqrt{(s - m.\omega^2)^2 + c^2.\omega^2}}$$ . . . (Same as before)

and $$\tan\phi = \frac{cd}{od} = \frac{c.\omega.A}{s.A - m.\omega^2.A} = \frac{c.\omega^2}{s - m.\omega^2}$$ . . . (Same as before)

23.17. Magnification Factor or Dynamic Magnifier

It is the ratio of ***maximum displacement of the forced vibration (x_{max}) to the deflection due to the static force $F(x_o)$.*** We have proved in the previous article that the maximum displacement or the amplitude of forced vibration,

$$x_{max} = \frac{x_o}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

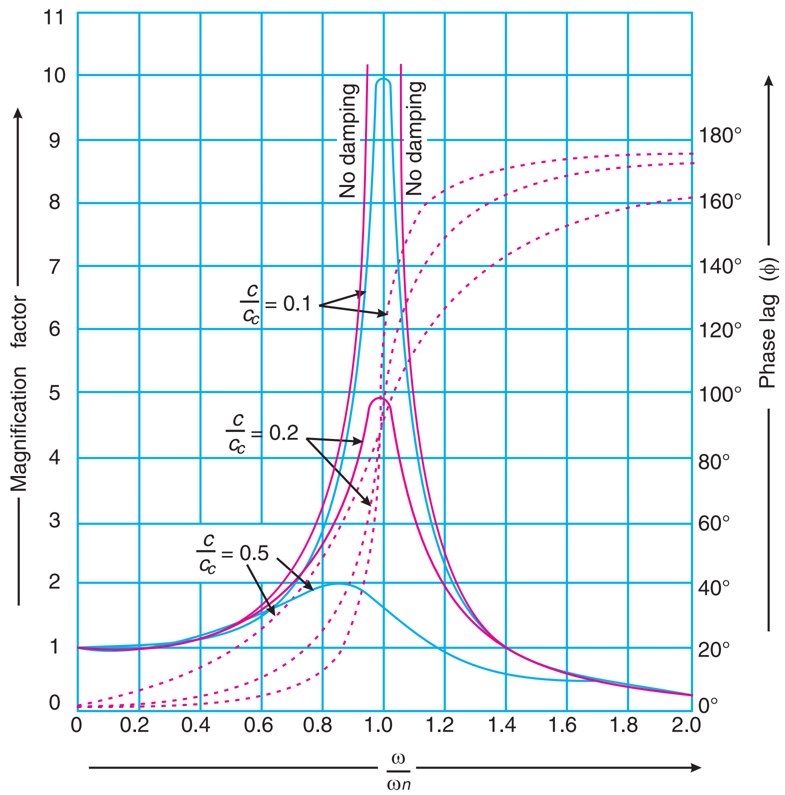


Fig. 23.21. Relationship between magnification factor and phase angle for different values of ω / ω_n .

∴ Magnification factor or dynamic magnifier,

$$D = \frac{x_{max}}{x_o} = \frac{1}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}} \qquad \text{... } (i)$$

$$= \frac{1}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n}\right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

$$\text{. . . } \left[\because \frac{c.\omega}{s} = \frac{2c.\omega}{2m \times \frac{s}{m}} = \frac{2c.\omega}{2m(\omega_n)^2} = \frac{2c.\omega}{c_c.\omega_n}\right]$$

The magnification factor or dynamic magnifier gives the factor by which the static deflection produced by a force F (*i.e.* x_o) must be multiplied in order to obtain the maximum amplitude of the forced vibration (*i.e.* x_{max}) by the harmonic force $F \cos \omega t$

$$\therefore \qquad x_{max} = x_o \times D$$

Fig. 23.21 shows the relationship between the magnification factor (D) and phase angle ϕ for different value of ω / ω_n and for values of damping factor c/c_c = 0.1, 0.2 and 0.5.

Notes: 1. If there is no damping (*i.e.* if the vibration is undamped), then c = 0. In that case, magnification factor,

$$D = \frac{x_{max}}{x_o} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}} = \frac{(\omega_n)^2}{(\omega_n)^2 - \omega^2}$$

2. At resonance, $\omega = \omega_n$. Therefore magnification factor,

$$D = \frac{x_{max}}{x_o} = \frac{s}{c.\omega_n}$$

Depending upon the case bridges can be treated as beams subjected to uniformly distributed leads and point loads.

Example 23.17. *A single cylinder vertical petrol engine of total mass 300 kg is mounted upon a steel chassis frame and causes a vertical static deflection of 2 mm. The reciprocating parts of the engine has a mass of 20 kg and move through a vertical stroke of 150 mm with simple harmonic motion. A dashpot is provided whose damping resistance is directly proportional to the velocity and amounts to 1.5 kN per metre per second.*

Considering that the steady state of vibration is reached ; determine : **1.** *the amplitude of forced vibrations, when the driving shaft of the engine rotates at 480 r.p.m., and* **2**. *the speed of the driving shaft at which resonance will occur.*

Solution : Given. m = 300 kg; δ = 2 mm = 2 × 10^{-3} m ; m_1 = 20 kg ; l = 150 mm = 0.15 m ; c = 1.5 kN/m/s = 1500 N/m/s ; N = 480 r.p.m. or $\omega = 2\pi \times 480 / 60$ = 50.3 rad/s

1. *Amplitude of the forced vibrations*

We know that stiffness of the frame,

$$s = m.g / \delta = 300 \times 9.81/2 \times 10^{-3} = 1.47 \times 10^6 \text{ N/m}$$

Since the length of stroke (l) = 150 mm = 0.15 m, therefore radius of crank,

$$r = l / 2 = 0.15 / 2 = 0.075 \text{ m}$$

We know that the centrifugal force due to the reciprocating parts or the static force,

$$F = m_1.\omega^2.r = 20\ (50.3)^2\ 0.075 = 3795 \text{ N}$$

∴ Amplitude of the forced vibration (maximum),

$$x_{max} = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}}$$

$$= \frac{3795}{\sqrt{(1500)^2 (50.3)^2 + [1.47 \times 10^6 - 300\,(50.3)^2]^2}}$$

$$= \frac{3795}{\sqrt{5.7 \times 10^9 + 500 \times 10^9}} = \frac{3795}{710 \times 10^3} = 5.3 \times 10^{-3} \text{ m}$$

= 5.3 mm **Ans.**

2. *Speed of the driving shaft at which the resonance occurs*

Let N = Speed of the driving shaft at which the resonance occurs in r.p.m.

We know that the angular speed at which the resonance occurs,

$$\omega = \omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{1.47 \times 10^6}{300}} = 70 \text{ rad/s}$$

∴ $N = \omega \times 60 / 2\pi = 70 \times 60 / 2\pi$ = 668.4 r.p.m. **Ans.**

Example 23.18. *A mass of 10 kg is suspended from one end of a helical spring, the other end being fixed. The stiffness of the spring is 10 N/mm. The viscous damping causes the amplitude to decrease to one-tenth of the initial value in four complete oscillations. If a periodic force of 150 cos 50 t N is applied at the mass in the vertical direction, find the amplitude of the forced vibrations. What is its value of resonance ?*

Solution. Given : m = 10 kg ; s = 10 N/mm = 10 × 10^3 N/m ; $x_5 = \frac{x_1}{10}$

Since the periodic force, $F_x = F \cos \omega t = 150 \cos 50 t$, therefore

Static force, F = 150 N

and angular velocity of the periodic disturbing force,

$$\omega = 50 \text{ rad/s}$$

We know that angular speed or natural circular frequency of free vibrations,

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{10 \times 10^3}{10}} = 31.6 \text{ rad/s}$$

Amplitude of the forced vibrations

Since the amplitude decreases to 1/10th of the initial value in four complete oscillations, therefore, the ratio of initial amplitude (x_1) to the final amplitude after four complete oscillations (x_5) is given by

$$\frac{x_1}{x_5} = \frac{x_1}{x_2} \times \frac{x_2}{x_3} \times \frac{x_3}{x_4} \times \frac{x_4}{x_5} = \left(\frac{x_1}{x_2}\right)^4 \qquad \ldots \left(\because \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5}\right)$$

$$\therefore \quad \frac{x_1}{x_2} = \left(\frac{x_1}{x_5}\right)^{1/4} = \left(\frac{x_1}{x_1/10}\right)^{1/4} = (10)^{1/4} = 1.78 \qquad \ldots \left(x_5 = \frac{x_1}{10}\right)$$

We know that

$$\log_e\left(\frac{x_1}{x_2}\right) = a \times \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

$$\log_e 1.78 = a \times \frac{2\pi}{\sqrt{(31.6)^2 - a^2}} \quad \text{or} \quad 0.576 = \frac{a \times 2\pi}{\sqrt{1000 - a^2}}$$

Squaring both sides and rearranging,

$$39.832\, a^2 = 332 \quad \text{or} \quad a^2 = 8.335 \quad \text{or} \quad a = 2.887$$

We know that $a = c/2m$ or $c = a \times 2m = 2.887 \times 2 \times 10 = 57.74$ N/m/s and deflection of the system produced by the static force F,

$$x_o = F/s = 150/10 \times 10^3 = 0.015 \text{ m}$$

We know that amplitude of the forced vibrations,

$$x_{max} = \frac{x_o}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left[1 - \frac{\omega^2}{(\omega_n)^2}\right]^2}}$$

$$= \frac{0.015}{\sqrt{\frac{(57.74)^2(50)^2}{(10 \times 10^3)^2} + \left[1 - \left(\frac{50}{31.6}\right)^2\right]^2}} = \frac{0.015}{\sqrt{0.083 + 2.25}}$$

$$= \frac{0.015}{1.53} = 9.8 \times 10^{-3} \text{ m} = 9.8 \text{ mm} \quad \textbf{Ans.}$$

Amplitude of forced vibrations at resonance

We know that amplitude of forced vibrations at resonance,

$$x_{max} = x_0 \times \frac{s}{c.\omega_n} = 0.015 \times \frac{10 \times 10^3}{57.54 \times 31.6} = 0.0822 \text{ m} = 82.2 \text{ mm} \quad \textbf{Ans.}$$

Example 23.19. *A body of mass 20 kg is suspended from a spring which deflects 15 mm under this load. Calculate the frequency of free vibrations and verify that a viscous damping force amounting to approximately 1000 N at a speed of 1 m/s is just-sufficient to make the motion aperiodic.*

If when damped to this extent, the body is subjected to a disturbing force with a maximum value of 125 N making 8 cycles/s, find the amplitude of the ultimate motion.

Solution . Given : m = 20 kg ; δ = 15 mm = 0.015 m ; c = 1000 N/m/s ; F = 125 N ; f = 8 cycles/s

Frequency of free vibrations

We know that frequency of free vibrations,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{1}{2\pi}\sqrt{\frac{9.81}{0.015}} = 4.07 \text{ Hz} \quad \textbf{Ans.}$$

The critical damping to make the motion aperiodic is such that damped frequency is zero, *i.e.*

$$\left(\frac{c}{2m}\right)^2 = \frac{s}{m}$$

$$\therefore \quad c = \sqrt{\frac{s}{m}\times 4m^2} = \sqrt{4\,s.m} = \sqrt{4\times\frac{m.g}{\delta}\times m} \quad \ldots \left(\because\ s = \frac{m.g}{\delta}\right)$$

$$= \sqrt{4\times\frac{20\times 9.81}{0.015}\times 20} = 1023 \text{ N/m/s}$$

This means that the viscous damping force is 1023 N at a speed of 1 m/s. Therefore a viscous damping force amounting to approximately 1000 N at a speed of 1 m/s is just sufficient to make the motion aperiodic. **Ans.**

Amplitude of ultimate motion

We know that angular speed of forced vibration,

$$\omega = 2\pi\times f = 2\pi\times 8 = 50.3 \text{ rad/s}$$

and stiffness of the spring, $s = m.g/\delta = 20 \times 9.81 / 0.015 = 13.1 \times 10^3$ N/m

$\therefore$ Amplitude of ultimate motion *i.e.* maximum amplitude of forced vibration,

$$x_{max} = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}}$$

$$= \frac{125}{\sqrt{(1023)^2(50.3)^2 + [13.1\times 10^3 - 20(50.3)^2]^2}}$$

$$= \frac{125}{\sqrt{2600\times 10^6 + 1406\times 10^6}} = \frac{125}{63.7\times 10^3} = 1.96 \times 10^{-3} \text{ m}$$

$= 1.96$ mm **Ans.**

Example 23.20. *A machine part of mass 2 kg vibrates in a viscous medium. Determine the damping coefficient when a harmonic exciting force of 25 N results in a resonant amplitude of 12.5 mm with a period of 0.2 second. If the system is excited by a harmonic force of frequency 4 Hz what will be the percentage increase in the amplitude of vibration when damper is removed as compared with that with damping.*

Solution . Given : m = 2 kg ; F = 25 N ; Resonant x_{max} = 12.5 mm = 0.0125 m ; t_p = 0.2 s ; f = 4 Hz

Damping coefficient

Let c = Damping coefficient in N/m/s.

We know that natural circular frequency of the exicting force,

$$\omega_n = 2\pi / t_p = 2\pi / 0.2 = 31.42 \text{ rad/s}$$

We also know that the maximum amplitude of vibration at resonance (x_{max}),

$$0.0125 = \frac{F}{c.\omega_n} = \frac{25}{c \times 31.42} = \frac{0.796}{c} \quad \text{or } c = 63.7 \text{ N/m/s } \textbf{Ans.}$$

Percentage increase in amplitude

Since the system is excited by a harmonic force of frequency (f) = 4 Hz, therefore corresponding circular frequency

$$\omega = 2\pi \times f = 2\pi \times 4 = 25.14 \text{ rad/s}$$

We know that maximum amplitude of vibration with damping,

$$x_{max} = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}}$$

$$= \frac{25}{\sqrt{(63.7)^2 (25.14)^2 + [2(31.42)^2 - 2(25.14)^2]^2}}$$

$$\ldots \left[\because (\omega_n)^2 = s/m \quad \text{or} \quad s = m(\omega_n)^2\right]$$

$$= \frac{25}{\sqrt{2.56 \times 10^6 + 0.5 \times 10^6}} = \frac{25}{1749} = 0.0143 \text{ m} = 14.3 \text{ mm}$$

and the maximum amplitude of vibration when damper is removed,

$$x_{max} = \frac{F}{m\left[(\omega_n)^2 - \omega^2\right]} = \frac{25}{2[(31.42)^2 - (25.14)^2]} = \frac{25}{710} = 0.0352 \text{ m}$$

$$= 35.2 \text{ mm}$$

∴ Percentage increase in amplitude

$$= \frac{35.2 - 14.3}{14.3} = 1.46 \quad \text{or} \quad 146\% \textbf{ Ans.}$$

Example 23.21. *The time of free vibration of a mass hung from the end of a helical spring is 0.8 second. When the mass is stationary, the upper end is made to move upwards with a displacement y metre such that y = 0.018 sin 2 π t, where t is the time in seconds measured from the beginning of the motion. Neglecting the mass of the spring and any damping effects, determine the vertical distance through which the mass is moved in the first 0.3 second.*

Solution. Given : t_p = 0.8 s ; y = 0.018 sin 2 π t

Let m = Mass hung to the spring in kg, and

s = Stiffness of the spring in N/m.

We know that time period of free vibrations (t_p),

$$0.8 = 2\pi\sqrt{\frac{m}{s}} \quad \text{or} \quad \frac{m}{s} = \left(\frac{0.8}{2\pi}\right)^2 = 0.0162$$

If x metres is the upward displacement of mass m from its equilibrium position after time t seconds, the equation of motion is given by

$$m \times \frac{d^2x}{dt^2} = s(y - x) \quad \text{or} \quad \frac{m}{s} \times \frac{d^2x}{dt^2} + x = y = 0.018 \sin 2\pi t$$

The solution of this differential equation is

$$x = A \sin \sqrt{\frac{s}{m}} \times t + B \cos \sqrt{\frac{s}{m}} \times t + \frac{0.018 \sin 2\pi t}{1 - \left(\frac{2\pi}{\sqrt{s/m}}\right)^2}$$

. . . (where A and B are constants)

$$= A \sin \frac{t}{\sqrt{0.0162}} + B \cos \frac{t}{\sqrt{0.0162}} + \frac{0.018 \sin 2\pi t}{1 - 4\pi^2 \times 0.0162}$$

$$= A \sin 7.85t + B \cos 7.85t + 0.05 \sin 2\pi t \quad \ldots (i)$$

Now when $t = 0$, $x = 0$, then from equation **(*i*)**, $B = 0$.

Again when $t = 0$, $dx/dt = 0$.

Therefore differentiating equation **(*i*)** and equating to zero, we have

$$dx/dt = 7.85 A \cos 7.85t + 0.05 \times 2\pi \cos 2\pi t = 0 \quad \ldots (\because B = 0)$$

or $$7.85 A \cos 7.85t = -0.05 \times 2\pi \cos 2\pi t$$

$\therefore$ $$A = -0.05 \times 2\pi / 7.85 = -0.04 \quad \ldots (\because t = 0)$$

Now the equation **(*i*)** becomes

$$x = -0.04 \sin 7.85t + 0.05 \sin 2\pi t \quad \ldots (\because B = 0) \ \ldots (ii)$$

$\therefore$ Vertical distance through which the mass is moved in the first 0.3 second (*i.e.* when $t = 0.3$ s),

$$= -0.04 \sin(7.85 \times 0.3) + 0.05 \sin(2\pi \times 0.3)$$

. . . [Substituting $t = 0.3$ in equation **(*ii*)**]

$$= -0.04 \times 0.708 + 0.05 \times 0.951 = -0.0283 + 0.0476 = 0.0193 \text{ m}$$

$= 19.3$ mm **Ans.**

23.18. Vibration Isolation and Transmissibility

A little consideration will show that when an unbalanced machine is installed on the foundation, it produces vibration in the foundation. In order to prevent these vibrations or to minimise the transmission of forces to the foundation, the machines are mounted on springs and dampers or on some vibration isolating material, as shown in Fig. 23.22. The arrangement is assumed to have one degree of freedom, *i.e.* it can move up and down only.

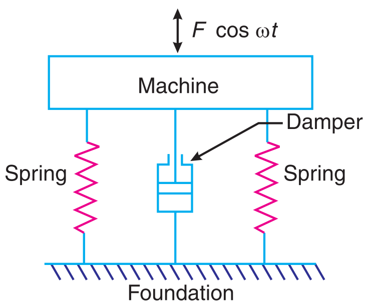


Fig. 23.22. Vibration isolation.

It may be noted that when a periodic (*i.e.* simple harmonic) disturbing force $F \cos \omega t$ is applied to a machine of mass m supported by a spring of stiffness s, then the force is transmitted by means of the spring and the damper or dashpot to the fixed support or foundation.

The ratio of the force transmitted (F_T) to the force applied (F) is known as the ***isolation factor*** or ***transmissibility ratio*** of the spring support.

We have discussed above that the force transmitted to the foundation consists of the following two forces :

1. Spring force or elastic force which is equal to $s.\,x_{max}$, and

2. Damping force which is equal to $c.\,\omega.x_{max}$.

Since these two forces are perpendicular to one another, as shown in Fig.23.23, therefore the force transmitted,

$$F_T = \sqrt{(s.x_{max})^2 + (c.\omega.x_{max})^2}$$

$$= x_{max}\sqrt{s^2 + c^2.\omega^2}$$

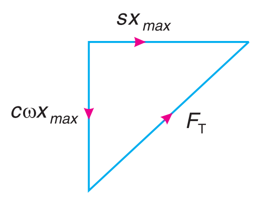


Fig. 23.23

∴ Transmissibility ratio,

$$\varepsilon = \frac{F_T}{F} = \frac{x_{max}\sqrt{s^2 + c^2.\omega^2}}{F}$$

We know that

$$x_{max} = x_o \times D = \frac{F}{s} \times D \qquad \ldots \left(\because\ x_o = \frac{F}{s}\right)$$

$$\therefore \qquad \varepsilon = \frac{D}{s}\sqrt{s^2 + c^2.\omega^2} = D\sqrt{1 + \frac{c^2.\omega^2}{s^2}}$$

$$= D\sqrt{1 + \left(\frac{2c}{c_c} \times \frac{\omega}{\omega_n}\right)^2} \qquad \ldots \left(\because\ \frac{c.\omega}{s} = \frac{2c}{c_c} \times \frac{\omega}{\omega_n}\right)$$

We have seen in Art. 23.17 that the magnification factor,

$$D = \frac{1}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n}\right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

$$\therefore \qquad \varepsilon = \frac{\sqrt{1 + \left(\frac{2c.\omega}{c_c.\omega_n}\right)^2}}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n}\right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}} \qquad \ldots (i)$$

When the damper is not provided, then $c = 0$, and

$$\varepsilon = \frac{1}{1 - (\omega/\omega_n)^2} \qquad \ldots (ii)$$

From above, we see that when $\omega/\omega_n > 1$, ε is negative. This means that there is a phase difference of 180° between the transmitted force and the disturbing force $(F \cos \omega t)$. The value of ω/ω_n must be greater than $\sqrt{2}$ if ε is to be less than 1 and it is the numerical value of ε, independent of any phase difference between the forces that may exist which is important. It is therefore more convenient to use equation **(*ii*)** in the following form, *i.e.*

$$\varepsilon = \frac{1}{(\omega/\omega_n)^2 - 1} \qquad \ldots (iii)$$

Fig. 23.24 is the graph for different values of damping factor c/c_c to show the variation of transmissibility ratio (ε) against the ratio ω/ω_n.

1. When $\omega/\omega_n = \sqrt{2}$, then all the curves pass through the point $\varepsilon = 1$ for all values of damping factor c/c_c.

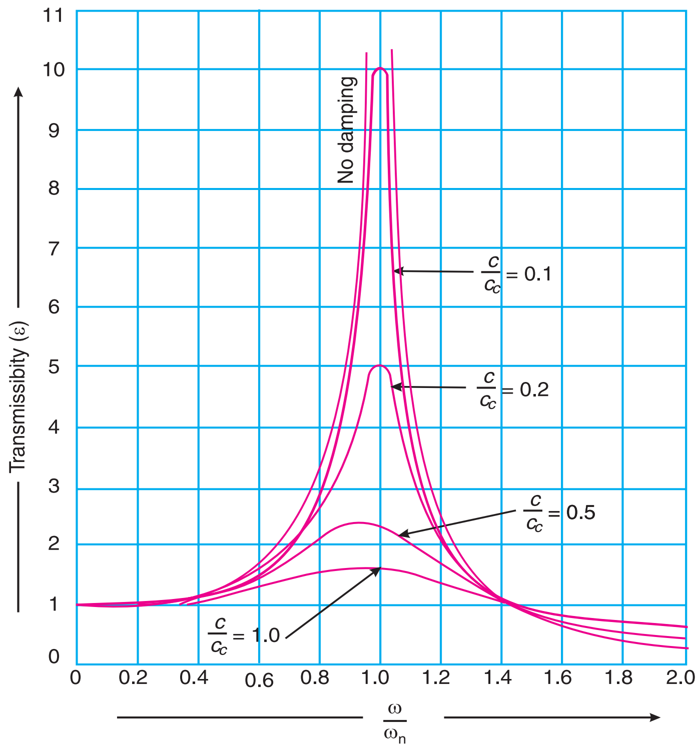


Fig. 23.24. Graph showing the variation of transmissibility ratio.

2. When $\omega/\omega_n < \sqrt{2}$, then $\varepsilon > 1$ for all values of damping factor c/c_c. This means that the force transmitted to the foundation through elastic support is greater than the force applied.

3. When $\omega/\omega_n > \sqrt{2}$, then $\varepsilon < 1$ for all values of damping factor c/c_c. This shows that the force transmitted through elastic support is less than the applied force. Thus vibration isolation is possible only in the range of $\omega/\omega_n > \sqrt{2}$.

We also see from the curves in Fig. 23.24 that the damping is detrimental beyond $\omega/\omega_n > \sqrt{2}$ and advantageous only in the region $\omega/\omega_n < \sqrt{2}$. It is thus concluded that for the vibration isolation, dampers need not to be provided but in order to limit resonance amplitude, stops may be provided.

Example 23.22. *The mass of an electric motor is 120 kg and it runs at 1500 r.p.m. The armature mass is 35 kg and its C.G. lies 0.5 mm from the axis of rotation. The motor is mounted on five springs of negligible damping so that the force transmitted is one-eleventh of the impressed force. Assume that the mass of the motor is equally distributed among the five springs.*

Determine **: 1.** *stiffness of each spring;* **2.** *dynamic force transmitted to the base at the operating speed; and* **3.** *natural frequency of the system.*

Solution. Given m_1 = 120 kg ; m_2 = 35 kg; r = 0.5 mm = 5 × 10^{-4} m; ε = 1 / 11; N = 1500 r.p.m. or ω = 2π × 1500 / 60 = 157.1 rad/s ;

1. *Stiffness of each spring*

Let s = Combined stiffness of the spring in N-m, and

ω_n = Natural circular frequency of vibration of the machine in rad/s.

We know that transmissibility ratio (ε),

$$\frac{1}{11} = \frac{1}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} = \frac{(\omega_n)^2}{\omega^2 - (\omega_n)^2} = \frac{(\omega_n)^2}{(157.1)^2 - (\omega_n)^2}$$

or $$(157.1)^2 - (\omega_n)^2 = 11(\omega_n)^2 \quad \text{or} \quad (\omega_n)^2 = 2057 \quad \text{or} \quad \omega_n = 45.35 \text{ rad/s}$$

We know that $$\omega_n = \sqrt{s/m_1}$$

$$s = m_1(\omega_n)^2 = 120 \times 2057 = 246\ 840 \text{ N/m}$$

Since these are five springs, therefore stiffness of each spring

= 246 840 / 5 = 49 368 N/m **Ans.**

2. *Dynamic force transmitted to the base at the operating speed (i.e. 1500 r.p.m. or 157.1 rad/s)*

We know that maximum unbalanced force on the motor due to armature mass,

$$F = m_2\, \omega^2 \cdot r = 35 (157.1)^2\, 5 \times 10^{-4} = 432 \text{ N}$$

∴ Dynamic force transmitted to the base,

$$F_T = \varepsilon . F = \frac{1}{11} \times 432 = 39.27 \text{ N}$$ **Ans.**

3. *Natural frequency of the system*

We have calculated above that the natural frequency of the system,

$$\omega_n = 45.35 \text{ rad/s}$$ **Ans.**

Example 23.23. *A machine has a mass of 100 kg and unbalanced reciprocating parts of mass 2 kg which move through a vertical stroke of 80 mm with simple harmonic motion. The machine is mounted on four springs, symmetrically arranged with respect to centre of mass, in such a way that the machine has one degree of freedom and can undergo vertical displacements only.*

Neglecting damping, calculate the combined stiffness of the spring in order that the force transmitted to the foundation is 1 / 25 th of the applied force, when the speed of rotation of machine crank shaft is 1000 r.p.m.

When the machine is actually supported on the springs, it is found that the damping reduces the amplitude of successive free vibrations by 25%. Find : **1.** *the force transmitted to foundation at 1000 r.p.m.,* **2.** *the force transmitted to the foundation at resonance, and* **3.** *the amplitude of the forced vibration of the machine at resonance.*

Solution. Given : m_1 = 100 kg ; m_2 = 2 kg ; l = 80 mm = 0.08 m ; ε = 1 / 25 ; N = 1000 r.p.m. or $\omega = 2\pi \times 1000/60$ = 104.7 rad/s

Combined stiffness of springs

Let s = Combined stiffness of springs in N/m, and

ω_n = Natural circular frequency of vibration of the machine in rad/s.

We know that transmissibility ratio (ε),

$$\frac{1}{25} = \frac{1}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} = \frac{(\omega_n)^2}{\omega^2 - (\omega_n)^2} = \frac{(\omega_n)^2}{(104.7)^2 - (\omega_n)^2}$$

or $\quad (104.7)^2 - (\omega_n)^2 = 25(\omega_n)^2 \quad$ or $\quad (\omega_n)^2 = 421.6$ or $\omega_n = 20.5$ rad/s

We know that $\quad \omega_n = \sqrt{s/m_1}$

$\therefore \quad s = m_1 (\omega_n)^2 = 100 \times 421.6$ = 42 160 N/m **Ans.**

1. Force transmitted to the foundation at 1000 r.p.m.

Let F_T = Force transmitted, and

x_1 = Initial amplitude of vibration.

Since the damping reduces the amplitude of successive free vibrations by 25%, therefore final amplitude of vibration,

$$x_2 = 0.75\, x_1$$

We know that

$$\log_e\left(\frac{x_1}{x_2}\right) = \frac{a \times 2\pi}{\sqrt{(\omega_n)^2 - a^2}} \quad \text{or} \quad \log_e\left(\frac{x_1}{0.75 x_1}\right) = \frac{a \times 2\pi}{\sqrt{421.6 - a^2}}$$

Squaring both sides,

$$(0.2877)^2 = \frac{a^2 \times 4\pi^2}{421.6 - a^2} \quad \text{or} \quad 0.083 = \frac{39.5\, a^2}{421.6 - a^2}$$

$$\ldots \left[\because \log_e\left(\frac{1}{0.75}\right) = \log_e 1.333 = 0.2877\right]$$

$$35 - 0.083\, a^2 = 39.5\, a^2 \quad \text{or} \quad a^2 = 0.884 \quad \text{or} \quad a = 0.94$$

We know that damping coefficient or damping force per unit velocity,

$$c = a \times 2m_1 = 0.94 \times 2 \times 100 = 188 \text{ N/m/s}$$

and critical damping coefficient,

$$c_c = 2m.\omega_n = 2 \times 100 \times 20.5 = 4100 \text{ N/m/s}$$

∴ Actual value of transmissibility ratio,

$$\varepsilon = \frac{\sqrt{1 + \left(\frac{2c.\omega}{c_c.\omega_n}\right)^2}}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n}\right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

$$= \frac{\sqrt{1 + \left(\frac{2 \times 188 \times 104.7}{4100 \times 20.5}\right)^2}}{\sqrt{\left(\frac{2 \times 188 \times 104.7}{4100 \times 20.5}\right)^2 + \left[1 - \left(\frac{104.7}{20.5}\right)^2\right]^2}} = \frac{\sqrt{1 + 0.22}}{\sqrt{0.22 + 629}}$$

$$= \frac{1.104}{25.08} = 0.044$$

We know that the maximum unbalanced force on the machine due to reciprocating parts,

$$F = m_2.\omega^2.r = 2(104.7)^2 (0.08/2) = 877 \text{ N} \qquad \ldots (\because r = l/2)$$

∴ Force transmitted to the foundation,

$$F_T = \varepsilon.F = 0.044 \times 877 = 38.6 \text{ N} \quad \textbf{Ans.} \qquad \ldots (\because \varepsilon = F_T / F)$$

2. Force transmitted to the foundation at resonance

Since at resonance, $\omega = \omega_n$, therefore transmissibility ratio,

$$\varepsilon = \frac{\sqrt{1 + \left(\frac{2c}{c_c}\right)^2}}{\sqrt{\left(\frac{2c}{c_c}\right)^2}} = \frac{\sqrt{1 + \left(\frac{2 \times 188}{4100}\right)^2}}{\sqrt{\left(\frac{2 \times 188}{4100}\right)^2}} = \frac{\sqrt{1 + 0.0084}}{0.092} = 10.92$$

and maximum unbalanced force on the machine due to reciprocating parts at resonance speed ω_n,

$$F = m_2 (\omega_n)^2 r = 2(20.5)^2 (0.08/2) = 33.6 \text{ N} \qquad \ldots (\because r = l/2)$$

∴ Force transmitted to the foundation at resonance,

$$F_T = \varepsilon.F = 10.92 \times 33.6 = 367 \text{ N} \quad \textbf{Ans.}$$

3. Amplitude of the forced vibration of the machine at resonance

We know that amplitude of the forced vibration at resonance

$$= \frac{\text{Force transmitted at resonance}}{\text{Combined stiffness}} = \frac{367}{42\,160} = 8.7 \times 10^{-3} \text{ m}$$

= 8.7 mm **Ans.**

Example 23.24. *A single-cylinder engine of total mass 200 kg is to be mounted on an elastic support which permits vibratory movement in vertical direction only. The mass of the piston is 3.5 kg and has a vertical reciprocating motion which may be assumed simple harmonic with a stroke of 150 mm. It is desired that the maximum vibratory force transmitted through the elastic support to the foundation shall be 600 N when the engine speed is 800 r.p.m. and less than this at all higher speeds.*

1. *Find the necessary stiffness of the elastic support, and the amplitude of vibration at 800 r.p.m., and*

2. *If the engine speed is reduced below 800 r.p.m. at what speed will the transmitted force again becomes 600 N?*

Solution. Given : m_1 = 200 kg ; m_2 = 3.5 kg ; l = 150 mm = 0.15 mm or $r = l/2$ = 0.075 m ; F_T = 600 N ; N = 800 r.p.m. or $\omega = 2\pi \times 800/60$ = 83.8 rad/s

We know that the disturbing force at 800 r.p.m.,

F = Centrifugal force on the piston

$$= m_2.\omega^2.r = 3.5\,(83.8)^2\,0.075 = 1843 \text{ N}$$

1. Stiffness of elastic support and amplitude of vibration

Let s = Stiffness of elastic support in N/m, and

x_{max} = Max. amplitude of vibration in metres.

Since the max. vibratory force transmitted to the foundation is equal to the force on the elastic support (neglecting damping), therefore

Max. vibratory force transmitted to the foundation,

F_T = Force on the elastic support

= Stiffness of elastic support × Max. amplitude of vibration

$$= s \times x_{max} = s \times \frac{F}{m\left[\omega^2 - (\omega_n)^2\right]}$$

$$= s \times \frac{F}{m\left(\omega^2 - \dfrac{s}{m}\right)} = \frac{F.s}{m.\omega^2 - s} \qquad \ldots \left[\because\ (\omega_n)^2 = \frac{s}{m}\right]$$

$$\therefore \qquad 600 = \frac{1843 \times s}{200\,(83.8)^2 - s} = \frac{1843\,s}{1.4 \times 10^6 - s} \qquad \ldots \text{(Substituting } m = m_1\text{)}$$

---

* The equation ***(x)*** of Art. 23.16 is

$$x_{max} = \frac{F}{m\left[(\omega_n)^2 - \omega^2\right]}$$

Since the max. vibratory force transmitted to the foundation through the elastic support decreases at all higher speeds (*i.e.* above N = 800 r.p.m. or ω = 83.8 rad/s), therefore we shall use

$$x_{max} = \frac{F}{m\left[\omega^2 - (\omega_n)^2\right]}$$

or $840 \times 10^6 - 600\, s = 1843\, s$

$\therefore$ $s = 0.344 \times 10^6 = 344 \times 10^3$ N/m **Ans.**

and maximum amplitude of vibration,

$$x_{max} = \frac{F}{m.\omega^2 - s} = \frac{1843}{200\,(83.8)^2 - 344\times10^3} = \frac{1843}{1056\times10^3} \text{ m}$$

$$= 1.745 \times 10^{-3} \text{ m} = 1.745 \text{ mm } \textbf{Ans.}$$

2. *Speed at the which the transmitted force again becomes 600 N*

The transmitted force will rise as the speed of the engine falls and passes through resonance. There will be a speed below resonance at which the transmitted force will again equal to 600 N. Let this speed be ω_1 rad/s (or N_1 r.p.m.).

$\therefore$ Disturbing force, $F = m_2\,(\omega_1)^2\, r = 3.5(\omega_1)^2\, 0.075 = 0.2625\,(\omega_1)^2$ N

Since the engine speed is reduced below N_1 = 800 r.p.m., therefore in this case, max, amplitude of vibration,

$$x_{max} = \frac{F}{m\left[(\omega_n)^2 - (\omega_1)^2\right]} = \frac{F}{m\left[\frac{s}{m} - (\omega_1)^2\right]} = \frac{F}{s - m\,(\omega_1)^2}$$

and Force transmitted $= s \times \dfrac{F}{s - m\,(\omega_1)^2}$

$$\therefore \quad 600 = 344\times10^3 \times \frac{0.2625\,(\omega_1)^2}{344\times10^3 - 200\,(\omega_1)^2} = \frac{90.3\times10^3\,(\omega_1)^2}{344\times10^2 - 200\,(\omega_1)^2}$$

. . . (Substituting $m = m_1$)

$206.4\times10^6 - 120\times10^3\,(\omega_1)^2 = 90.3\times10^3\,(\omega_1)^2$ or $(\omega_1)^2 = 981$

$\therefore$ $\omega_1 = 31.32$ rad/s or $N_1 = 31.32 \times 60 / 2\pi$ = 299 r.p.m. **Ans.**

EXERCISES

1. A shaft of 100 mm diameter and 1 metre long is fixed at one end and other end carries a flywheel of mass 1 tonne. Taking Young's modulus for the shaft material as 200 GN/m^2, find the natural frequency of longitudinal and transverse vibrations. **[Ans. 200 Hz ; 8.6 Hz]**

2. A beam of length 10 m carries two loads of mass 200 kg at distances of 3 m from each end together with a central load of mass 1000 kg. Calculate the frequency of transverse vibrations. Neglect the mass of the beam and take $I = 10^9$ mm^4 and $E = 205\times10^3$ N/mm^2. **[Ans. 13.8 Hz]**

3. A steel bar 25 mm wide and 50 mm deep is freely supported at two points 1 m apart and carries a mass of 200 kg in the middle of the bar. Neglecting the mass of the bar, find the frequency of transverse vibration.

If an additional mass of 200 kg is distributed uniformly over the length of the shaft, what will be the frequency of vibration ? Take E = 200 GN/m^2. **[Ans. 17.8 Hz ; 14.6 Hz]**

4. A shaft 1.5 m long is supported in flexible bearings at the ends and carries two wheels each of 50 kg mass. One wheel is situated at the centre of the shaft and the other at a distance of 0.4 m from the centre towards right. The shaft is hollow of external diameter 75 mm and inner diameter 37.5 mm. The density of the shaft material is 8000 kg/m^3. The Young's modulus for the shaft material is 200 GN/m^2. Find the frequency of transverse vibration. **[Ans. 33.2 Hz]**

5. A shaft of diameter 10 mm carries at its centre a mass of 12 kg. It is supported by two short bearings, the centre distance of which is 400 mm. Find the whirling speed : 1. neglecting the mass of the shaft, and 2. taking the mass of the shaft also into consideration. The density of shaft material is 7500 kg/m^3. **[Ans. 748 r.p.m.; 744 r.p.m.]**

6. A shaft 180 mm diameter is supported in two bearings 2.5 metres apart. It carries three discs of mass 250 kg, 500 kg and 200 kg at 0.6 m, 1.5 m and 2 m from the left hand. Assuming the mass of the shaft 190 kg/m, determine the critical speed of the shaft. Young's modulus for the material of the shaft is 211 GN/m^2. **[Ans. 18.8 r.p.m.]**

7. A shaft 12.5 mm diameter rotates in long bearings and a disc of mass 16 kg is secured to a shaft at the middle of its length. The span of the shaft between the bearing is 0.5 m. The mass centre of the disc is 0.5 mm from the axis of the shaft. Neglecting the mass of the shaft and taking E = 200 GN/m^2, find : 1 critical speed of rotation in r.p.m., and 2. the range of speed over which the stress in the shaft due to bending will not exceed 120 MN/m^2. Take the static deflection of the shaft for a beam fixed at both ends, *i.e.* $\delta = \dfrac{Wl^3}{192\,EI}$. **[Ans. 1450 r.p.m. ; 1184 to 2050 r.p.m.]**

8. A vertical shaft 25 mm diameter and 0.75 m long is mounted in long bearings and carries a pulley of mass 10 kg midway between the bearings. The centre of pulley is 0.5 mm from the axis of the shaft. Find (*a*) the whirling speed, and (*b*) the bending stress in the shaft, when it is rotating at 1700 r.p.m. Neglect the mass of the shaft and E = 200 GN/m^2. **[Ans. 3996 r.p.m ; 12.1 MN/m^2]**

9. A shaft 12 mm in diameter and 600 mm long between long bearings carries a central mass of 4 kg. If the centre of gravity of the mass is 0.2 mm from the axis of the shaft, compute the maximum flexural stress in the shaft when it is running at 90 per cent of its critical speed. The value of Young's modulus of the material of the shaft is 200 GN/m^2. **[Ans. 14.8 kN/m^2]**

10. A vibrating system consists of a mass of 8 kg, spring of stiffness 5.6 N/mm and a dashpot of damping coefficient of 40 N/m/s. Find (*a*) damping factor, (*b*) logarithmic decrement, and (*c*) ratio of the two consecutive amplitudes. **[Ans. 0.094 ; 0.6 ; 1.82]**

11. A body of mass of 50 kg is supported by an elastic structure of stiffness 10 kN/m. The motion of the body is controlled by a dashpot such that the amplitude of vibration decreases to one-tenth of its original value after two complete vibrations. Determine : 1. the damping force at 1 m/s ; 2. the damping ratio, and 3. the natural frequency of vibration. **[Ans. 252 N/m/s ; 0.178 ; 2.214 Hz]**

12. A mass of 85 kg is supported on springs which deflect 18 mm under the weight of the mass. The vibrations of the mass are constrained to be linear and vertical and are damped by a dashpot which reduces the amplitude to one quarter of its initial value in two complete oscillations. Find : 1. the magnitude of the damping force at unit speed, and 2. the periodic time of damped vibration. **[Ans. 435 N/m/s ; 0.27 s]**

13. The mass of a machine is 100 kg. Its vibrations are damped by a viscous dash pot which diminishes amplitude of vibrations from 40 mm to 10 mm in three complete oscillations. If the machine is mounted on four springs each of stiffness 25 kN/m, find (*a*) the resistance of the dash pot at unit velocity, and (*b*) the periodic time of the damped vibration. **[Ans. 6.92 N/m/s ; 0.2 s]**

14. A mass of 7.5 kg hangs from a spring and makes damped oscillations. The time for 60 oscillations is 35 seconds and the ratio of the first and seventh displacement is 2.5. Find (*a*) the stiffness of the spring, and (*b*) the damping resistance in N/m/s. If the oscillations are critically damped, what is the damping resistance required in N/m/s ? **[Ans. 870 N/m ; 3.9 N/m/s ; 162 N/m/s]**

15. A mass of 5 kg is supported by a spring of stiffness 5 kN/m. In addition, the motion of mass is controlled by a damper whose resistance is proportional to velocity. The amplitude of vibration reduces to 1/15th of the initial amplitude in four complete cycles. Determine the damping force per unit velocity and the ratio of the frequencies of the damped and undamped vibrations. **[Ans. 34 N/m/s : 0.994]**

16. A mass of 50 kg suspended from a spring produces a statical deflection of 17 mm and when in motion it experiences a viscous damping force of value 250 N at a velocity of 0.3 m/s. Calculate the periodic time of damped vibration. If the mass is then subjected to a periodic disturbing force having a maximum value of 200 N and making 2 cycles/s, find the amplitude of ultimate motion. **[Ans. 0.262 s ; 8.53 mm]**

17. A mass of 50 kg is supported by an elastic structure of total stiffness 20 kN/m. The damping ratio of the system is 0.2. A simple harmonic disturbing force acts on the mass and at any time *t* seconds, the force is 60 cos 10 *t* newtons. Find the amplitude of the vibrations and the phase angle caused by the damping. **[Ans. 3.865 mm ; 14.93°]**

18. A machine of mass 100 kg is supported on openings of total stiffness 800 kN/m and has a rotating unbalanced element which results in a disturbing force of 400 N at a speed of 3000 r.p.m. Assuming the damping ratio as 0.25, determine : 1. the amplitude of vibrations due to unbalance ; and 2. the transmitted force. **[Ans. 0.04 mm ; 35.2 N]**

19. A mass of 500 kg is mounted on supports having a total stiffness of 100 kN/m and which provides viscous damping, the damping ratio being 0.4. The mass is constrained to move vertically and is subjected to a vertical disturbing force of the type *F* cos ωt. Determine the frequency at which resonance will occur and the maximum allowable value of *F* if the amplitude at resonance is to be restricted to 5 mm. **[Ans. 2.25 Hz ; 400 N]**

20. A machine of mass 75 kg is mounted on springs of stiffness 1200 kN/m and with an assumed damping factor of 0.2. A piston within the machine of mass 2 kg has a reciprocating motion with a stroke of 80 mm and a speed of 3000 cycles/min. Assuming the motion to be simple harmonic, find : 1. the amplitude of motion of the machine, 2. its phase angle with respect to the exciting force, 3. the force transmitted to the foundation, and 4. the phase angle of transmitted force with respect to the exciting force. **[Ans. 1.254 mm ; 169.05° ; 2132 N ; 44.8°]**

DO YOU KNOW ?

1. What are the causes and effects of vibrations ?
2. Define, in short, free vibrations, forced vibrations and damped vibrations.
3. Discuss briefly with neat sketches the longitudinal, transverse and torsional free vibrations.
4. Derive an expression for the natural frequency of free transverse and longitudinal vibrations by equilibrium method.
5. Discuss the effect of inertia of the shaft in longitudinal and transverse vibrations.
6. Deduce an expression for the natural frequency of free transverse vibrations for a simply supported shaft carrying uniformly distributed mass of *m* kg per unit length.
7. Deduce an expression for the natural frequency of free transverse vibrations for a beam fixed at both ends and carrying a uniformly distributed mass of *m* kg per unit length.
8. Establish an expression for the natural frequency of free transverse vibrations for a simply supported beam carrying a number of point loads, by (*a*) Energy method ; and (*b*) Dunkerley's method.
9. Explain the term 'whirling speed' or 'critical speed' of a shaft. Prove that the whirling speed for a rotating shaft is the same as the frequency of natural transverse vibration.
10. Derive the differential equation characterising the motion of an oscillation system subject to viscous damping and no periodic external force. Assuming the solution to the equation, find the frequency of oscillation of the system.
11. Explain the terms 'under damping, critical damping' and 'over damping'
12. A thin plate of area *A* and mass *m* is attached to the end of a spring and is allowed to oscillate in a viscous fluid, as shown in Fig. 23.25. Show that

$$\mu = \frac{m}{A}\sqrt{\omega^2 - (\omega_d)^2}$$

where the damping force on the plate is equal to $\mu . A v$; *v* being the velocity.

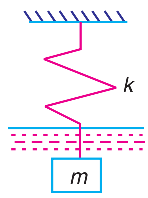


Fig. 23.25

The symbols ω and ω_d indicate the undamped and damped natural circular frequencies of oscillations.

13. Explain the term 'Logarithmic decrement' as applied to damped vibrations.
14. Establish an expression for the amplitude of forced vibrations.
15. Explain the term 'dynamic magnifier'.
16. What do you understand by transmissibility ?

OBJECTIVE TYPE QUESTIONS

1. When there is a reduction in amplitude over every cycle of vibration, then the body is said to have

(*a*) free vibration (*b*) forced vibration (*c*) damped vibration

2. Longitudinal vibrations are said to occur when the particles of a body moves

(*a*) perpendicular to its axis (*b*) parallel to its axis

(*c*) in a circle about its axis

3. When a body is subjected to transverse vibrations, the stress induced in a body will be

(*a*) shear stress (*b*) tensile stress (*c*) compressive stress

4. The natural frequency (in Hz) of free longitudinal vibrations is equal to

(*a*) $\frac{1}{2\pi}\sqrt{\frac{s}{m}}$ (*b*) $\frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$ (*c*) $\frac{0.4985}{\sqrt{\delta}}$

(*d*) any one of these

where m = Mass of the body in kg,

s = Stiffness of the body in N/m, and

δ = Static deflection of the body in metres.

5. The factor which affects the critical speed of a shaft is

(*a*) diameter of the disc (*b*) span of the shaft

(*c*) eccentricity (*d*) all of these

6. The equation of motion for a vibrating system with viscous damping is

$$\frac{d^2x}{dt^2} + \frac{c}{m} \times \frac{dx}{dt} + \frac{s}{m} \times x = 0$$

If the roots of this equation are real, then the system will be

(*a*) over damped (*b*) under damped (*c*) critically damped

7. In under damped vibrating system, if x_1 and x_2 are the successive values of the amplitude on the same side of the mean position, then the logarithmic decrement is equal to

(*a*) x_1/x_2 (*b*) $\log (x_1/x_2)$ (*c*) $\log_e (x_1/x_2)$ (*d*) $\log (x_1.x_2)$

8. The ratio of the maximum displacement of the forced vibration to the deflection due to the static force, is known as

(*a*) damping factor (*b*) damping coefficient

(*c*) logarithmic decrement (*d*) magnification factor

9. In vibration isolation system, if ω/ω_n is less than $\sqrt{2}$, then for all values of the damping factor, the transmissibility will be

(*a*) less than unity (*b*) equal to unity (*c*) greater than unity (*d*) zero

where ω = Circular frequency of the system in rad/s, and

ω_n = Natural circular frequency of vibration of the system in rad/s.

10. In vibration isolation system, if $\omega/\omega_n > 1$, then the phase difference between the transmitted force and the disturbing force is

(*a*) 0° (*b*) 90° (*c*) 180° (*d*) 270°

ANSWERS

1. (*c*)	**2.** (*b*)	**3.** (*b*)	**4.** (*d*)	**5.** (*d*)
6. (*a*)	**7.** (*b*)	**8.** (*d*)	**9.** (*c*)	**10.** (*c*)

GO To FIRST